Reconstruction of piecewise constant images via total variation regularization

exact support recovery and grid-free numerical methods

Romain Petit

PhD defense, December 12th 2022

Setting

Unknown image $u_0 \in \mathrm{L}^2(\mathbb{R}^2)$

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Observations $y_0 = \Phi u_0 \in \mathcal{H}$

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Inverse problem

Recover u_0 from y

Variational approach [Rudin et al., 1992, Chambolle and Lions, 1997]

The total (gradient) variation

$$\mathrm{TV}(u) \overset{\text{def.}}{=} \sup \left\{ -\int_{\mathbb{R}^2} u \,\mathrm{div}\phi \;\middle|\; \phi \in \mathrm{C}_c^\infty(\mathbb{R}^2, \mathbb{R}^2),\ \|\phi\|_\infty \leq 1 \right\} = |\mathrm{D}u|(\mathbb{R}^2)$$

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Solve

$$\min_{u \in \mathrm{L}^2(\mathbb{R}^2)} \frac{1}{2}\|\Phi u - y\|_{\mathcal{H}}^2 + \lambda \,\mathrm{TV}(u) \qquad (\mathcal{P}_\lambda(y))$$

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solution (well-chosen λ)

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noisy observations y | solution (well-chosen λ) | solution (high λ)

Solve

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Variational approach [Rudin et al., 1992, Chambolle and Lions, 1997]

Representer th. [Boyer et al., 2019, Bredies and Carioni, 2019] + [Fleming, 1957]

If $\dim(\mathcal{H}) < +\infty$, some sol. of $(\mathcal{P}_\lambda(y))$ are linear comb. of $\mathbf{1}_E$ with E *simple*

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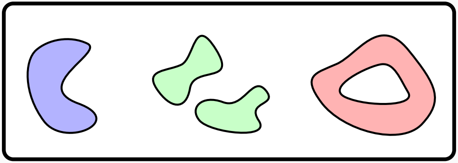

Solve

$$\min_{u \in \mathrm{L}^2(\mathbb{R}^2)} \frac{1}{2}\|\Phi u - y\|_{\mathcal{H}}^2 + \lambda\,\mathrm{TV}(u) \qquad (\mathcal{P}_\lambda(y))$$

Contributions

Considered problems

unknown im. u_0

obs. $y_0 = \Phi u_0$

noisy obs. $y = y_0 + w$

Contributions

Considered problems

- Convergence of solutions of $(\mathcal{P}_\lambda(y_0 + w))$ when $\lambda, w \to 0$

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- Convergence of solutions of $(\mathcal{P}_\lambda(y_0 + w))$ when $\lambda, w \to 0$
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Analysis of TV reg. from a “sparse recovery” viewpoint

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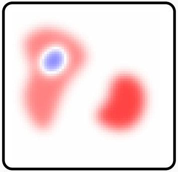

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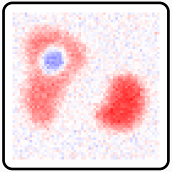

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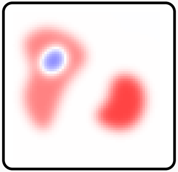

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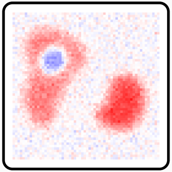

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Analysis of TV reg. from a "sparse recovery" viewpoint

- Assume u_0 is piecewise constant
- Structure-preserving stability results and numerical methods

Contributions

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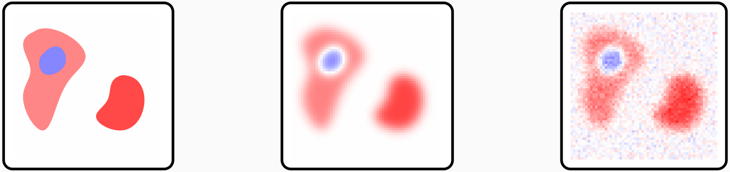
- Numerical resolution of $(\mathcal{P}_\lambda(y))$



unknown im. u_0 | obs. $y_0 = \Phi u_0$ | noisy obs. $y = y_0 + w$

Use of shape optimization tools

Contributions

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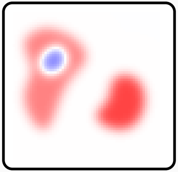

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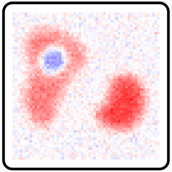

noisy obs. $y = y_0 + w$

Use of shape optimization tools

- Stability via second order shape derivatives

Contributions

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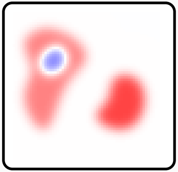

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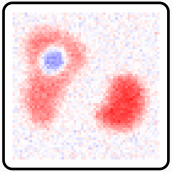

noisy obs. $y = y_0 + w$

Use of shape optimization tools

- Stability via second order shape derivatives
- Shape gradient algorithm

Noise robustness:
exact support recovery

State of the art (short) summary

$(\mathcal{P}_\lambda(\boldsymbol{y_0} + \boldsymbol{w}))$

$$\min_{u \in \mathrm{L}^2(\mathbb{R}^2)} \mathrm{TV}(u) + \frac{1}{2\lambda}\|\Phi u - (y_0 + w)\|^2$$

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$(\mathcal{P}_0(y_0))$

$$\min_{u \in \mathrm{L}^2(\mathbb{R}^2)} \mathrm{TV}(u) \text{ s.t. } \Phi u = y_0$$

State of the art (short) summary

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Proposition [Hofmann et al., 2007, Chambolle et al., 2016, Iglesias et al., 2018]

If $\lambda_n \to 0$, $\frac{\|w_n\|}{\lambda_n} \leq \frac{\sqrt{\pi}}{2\|\Phi^*\|}$ + source cond. then (up to extr.) $u_n \to u_0$ strictly in $\mathrm{BV}(\mathbb{R}^2)$

and for a.e. $t \in \mathbb{R}$, $\partial U_n^{(t)} \xrightarrow{\text{Hausdorff}} \partial U_0^{(t)}$ with $U^{(t)} = \begin{cases} \{u \geq t\} & \text{if } t \geq 0 \\ \{u \leq t\} & \text{otherwise.} \end{cases}$

State of the art (short) summary

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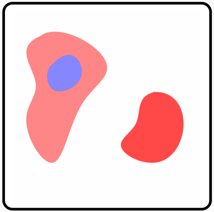
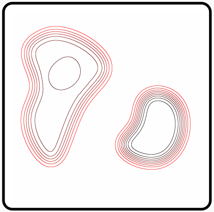


u_0

u_n



$U_n^{(t)}, \ t \in \mathbb{R}$

The prescribed curvature problem

Optimality condition (regularized pb.)

If $u \in \mathrm{L}^2(\mathbb{R}^2)$ solves $(\mathcal{P}_\lambda(y_0 + w))$ then

$$u \in \underset{v \in \mathrm{L}^2(\mathbb{R}^2)}{\text{Argmin}} \ \mathrm{TV}(v) - \int_{\mathbb{R}^2} \eta_{\lambda,w} \, v$$

with $\eta_{\lambda,w} = \Phi^* p_{\lambda,w}$ and $p_{\lambda,w}$ unique sol. of $(\mathcal{D}_\lambda(y_0 + w))$

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If $u \in \mathrm{L}^2(\mathbb{R}^2)$ solves $(\mathcal{P}_\lambda(y_0 + w))$ then

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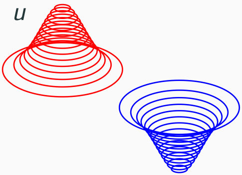

The prescribed curvature problem

Prescribed curvature problem

$$\min_{E\subset\mathbb{R}^2,\,|E|<+\infty} \mathrm{Per}(E) - \int_E \eta \qquad (\mathcal{Q}(\eta))$$

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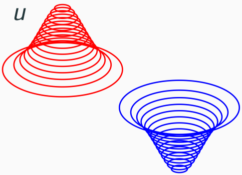

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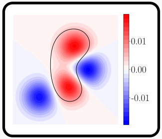

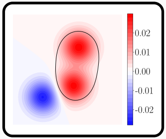

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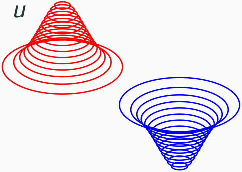

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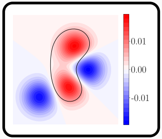

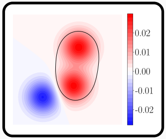

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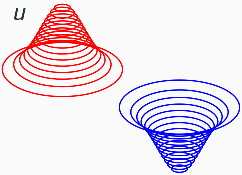

Convergence of curvature functionals

If $\lambda_n \to 0$ and $\frac{\|w_n\|}{\lambda_n} \to 0$ (+ source cond.) then $\eta_{\lambda_n,w_n} \to \eta_0$ strongly in $\mathrm{L}^2(\mathbb{R}^2)$

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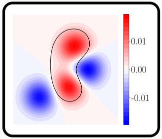

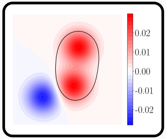

Optimality condition (constrained pb.)

If $u \in \mathrm{L}^2(\mathbb{R}^2)$ solves $(\mathcal{P}_0(y_0))$ then

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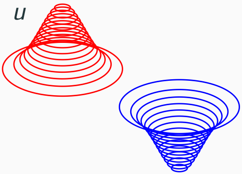

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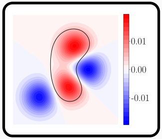

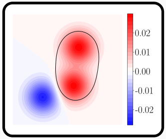

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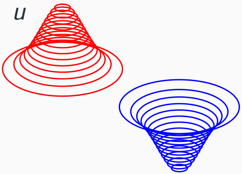

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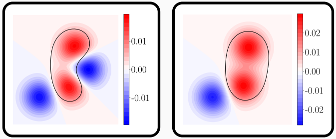

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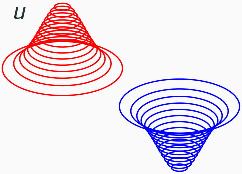

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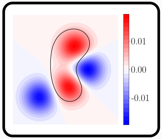
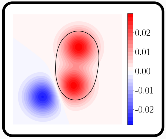
If $\lambda_n \to 0$ and $\frac{\|w_n\|}{\lambda_n} \to 0$ (+ source cond.) then $\eta_{\lambda_n,w_n} \to \eta_0$ strongly in $\mathrm{L}^2(\mathbb{R}^2)$

The prescribed curvature problem

Prescribed curvature problem

$$\min_{E\subset\mathbb{R}^2,\,|E|<+\infty} \operatorname{Per}(E) - \int_E \eta \qquad (\mathcal{Q}(\eta))$$





Our curvature functional

$$\eta \in \operatorname{Im}(\Phi^*) \subset \mathrm{L}^2(\mathbb{R}^2)$$

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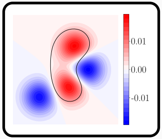

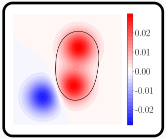

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Exploit regularity assumptions on Φ

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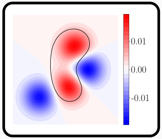

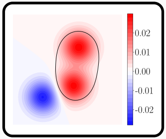

Our curvature functional

$$\eta \in \mathrm{Im}(\Phi^*) \subset \mathrm{L}^2(\mathbb{R}^2)$$

If $\eta \in \mathrm{C}^1_b(\mathbb{R}^2)$, every solution of $(\mathcal{Q}(\eta))$ is C^3

Convergence of minimizers

Assumptions

- $(\eta_n)_{n\geq 0}$ and η in $\mathrm{C}^1_b(\mathbb{R}^2)$
- $(\eta_n)_{n\geq 0}$ converges in $\mathrm{L}^2(\mathbb{R}^2)$ and $\mathrm{C}^1_b(\mathbb{R}^2)$ to η

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Proposition

For every $\epsilon > 0$ there exists n_0 such that for $n \geq n_0$ every sol. E_n of $(\mathcal{Q}(\eta_n))$ satisfies

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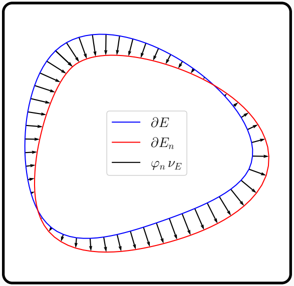
with E sol. of $(\mathcal{Q}(\eta))$ and $\|\varphi_n\|_{\mathrm{C}^2(\partial E)} \leq \epsilon$



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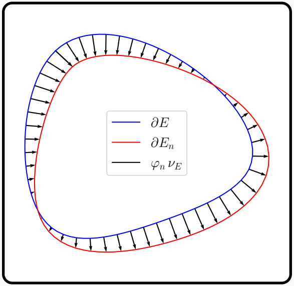

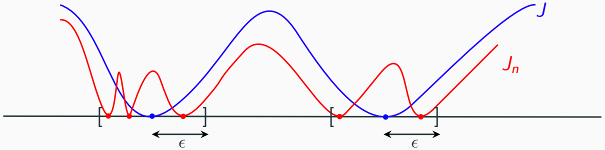

Stability via second variation

$$J(E) \stackrel{\text{def}}{=} \mathrm{Per}(E) - \int_E \eta \text{ with } \eta \in \mathrm{C}^1_b(\mathbb{R}^2)$$

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Local deformations

Behaviour around 0 of the mapping

$$\begin{aligned} \mathrm{j}_E : \mathrm{W}^{1,\infty}(\partial E) &\to \mathbb{R} \\ \varphi &\mapsto J(E_\varphi) \end{aligned}$$

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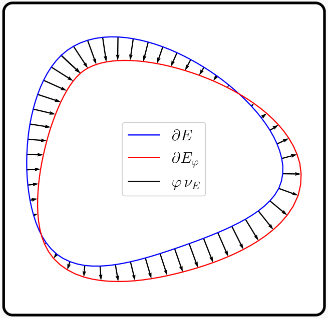

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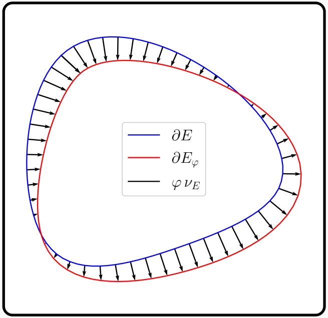

Definition

A critical point E of J is said to be strictly stable if

$$\forall \psi \in \mathrm{H}^1(\partial E),\ \mathrm{j}''_E(0).(\psi, \psi) > 0$$

$$J(E) \overset{\text{def}}{=} \mathrm{Per}(E) - \int_E \eta \text{ with } \eta \in \mathrm{C}^1_b(\mathbb{R}^2)$$

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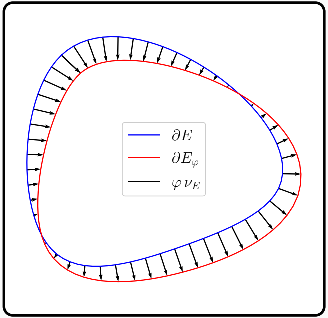

Proposition [Dambrine and Lamboley, 2019]

If E strictly stable minimizer of J, there exists $r > 0$ and $c > 0$ s.t.

$$\forall \varphi \text{ s.t. } \|\varphi\|_{\mathrm{W}^{1,\infty}(\partial E)} \leq r,\ J(E_\varphi) \geq J(E) + c\,\|\varphi\|^2_{\mathrm{H}^1(\partial E)}$$

Stability result

Assumptions

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There exists $\epsilon > 0$ s.t. for n large there is at most one φ_n s.t. $\|\varphi_n\|_{\mathrm{C}^2(\partial E)} \leq \epsilon$ and E_{φ_n} is a solution of $(\mathcal{Q}(\eta_n))$

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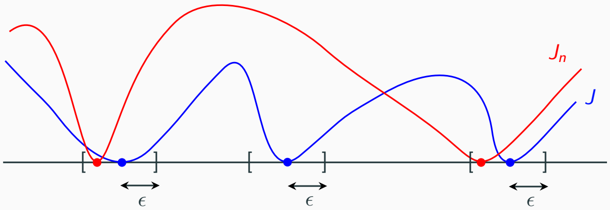

Support recovery (simplest case)

Assumptions

- $u_0 = a\,\mathbf{1}_E$, E simple, unique (non trivial) sol. of $(\mathcal{Q}(\eta_0))$, strict. stable
- $\mathrm{Im}(\Phi^*) \subset \mathrm{C}^1_b(\mathbb{R}^2)$
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Theorem

If $\lambda_n \to 0$ and $\frac{\|w_n\|}{\lambda_n} \to 0$, for n large enough $u_n = a_n\,\mathbf{1}_{E_n}$ with

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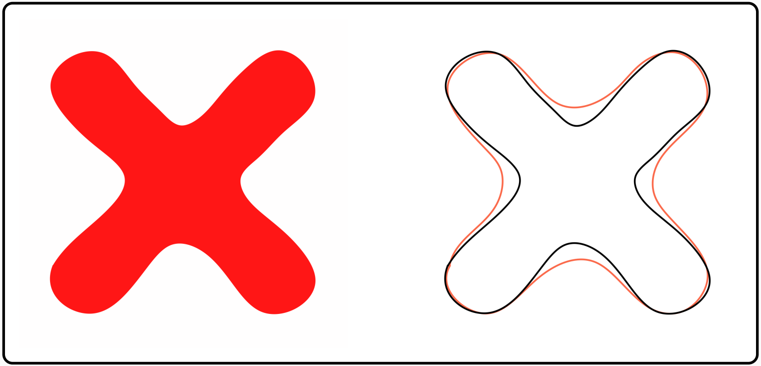

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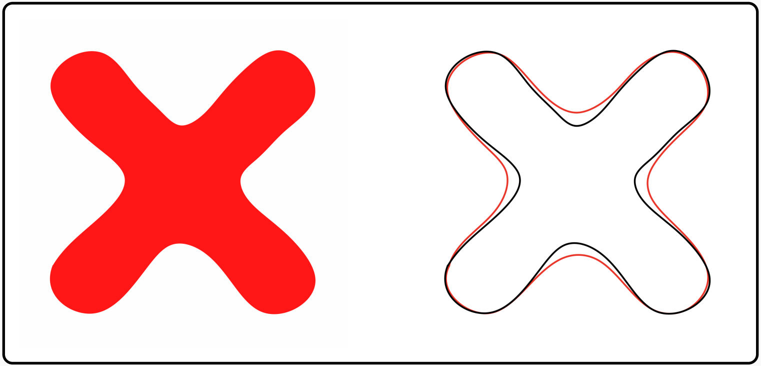

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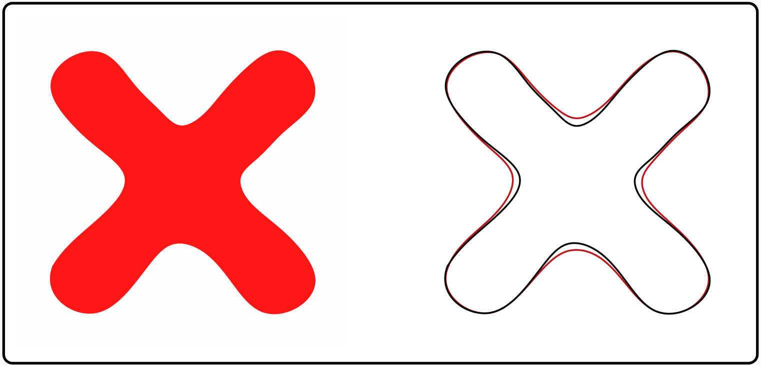

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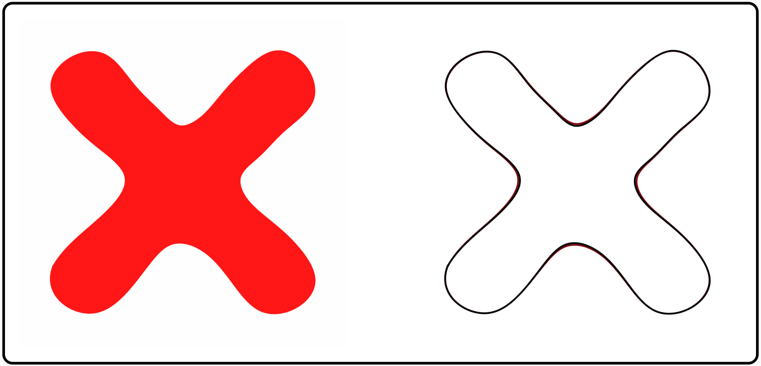

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Faces and chains

Optimality condition

If u solves $(\mathcal{P}_{\lambda,w}(y_0 + w))$ then

$$u \in \underset{v \in \mathrm{L}^2(\mathbb{R}^2)}{\mathrm{Argmin}} \ \mathrm{TV}(v) - \int_{\mathbb{R}^2} \eta_{\lambda,w}\, v$$

Faces and chains

Optimality condition

If $u \neq 0$ solves $(\mathcal{P}_{\lambda,w}(y_0 + w))$ then

$$\frac{u}{\mathrm{TV}(u)} \in \mathcal{F} = \underset{v \in \{\mathrm{TV} \leq 1\}}{\mathrm{Argmax}} \int_{\mathbb{R}^2} \eta_{\lambda,w}\, v$$

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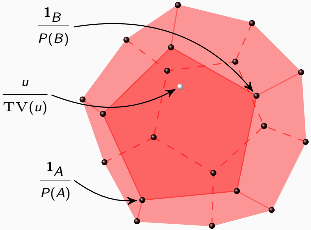

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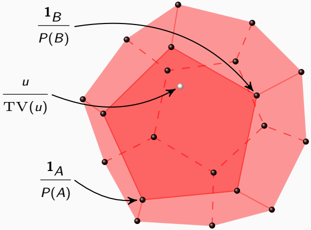

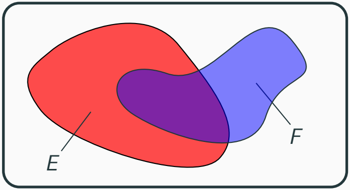

$$u = 2\,\mathbf{1}_E + \mathbf{1}_F$$

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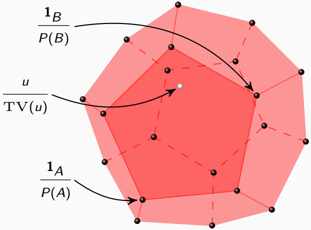

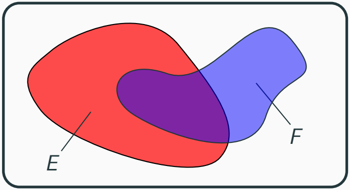

$$u = 3\,\mathbf{1}_{E\cap F} + 2\,\mathbf{1}_{E\setminus F} + \mathbf{1}_{F\setminus E}$$

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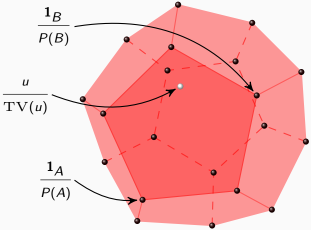

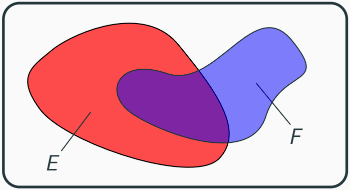

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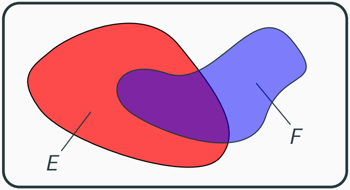
Chain

Increasing collection of sets whose indicator function belongs to $\mathcal{F}$



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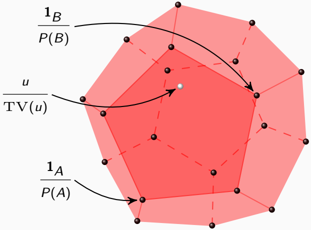

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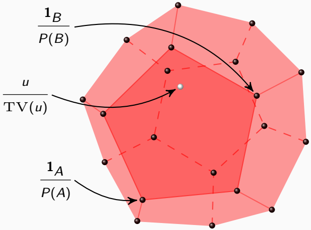

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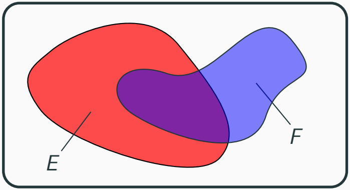

$u = 2\,\mathbf{1}_E + \mathbf{1}_F$

if $t > 3$, $\{u \geq t\} = \emptyset$

Faces and chains

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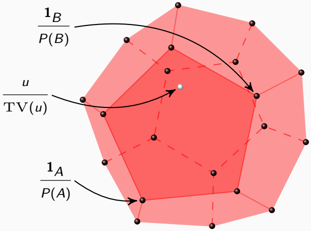

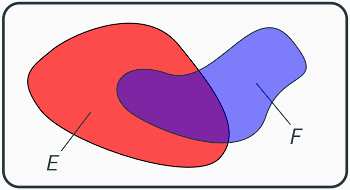

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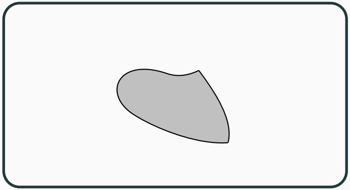

$\{u \geq 3\} = E \cap F$

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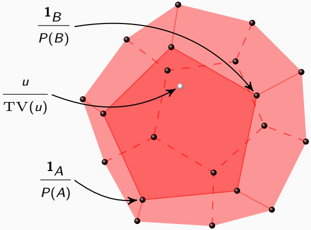

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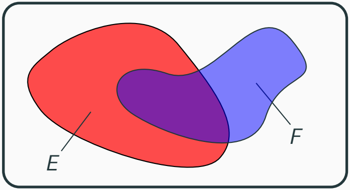

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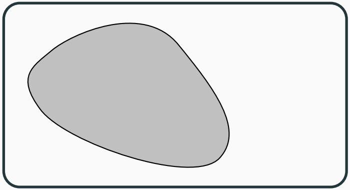

$\{u \geq 2\} = E$

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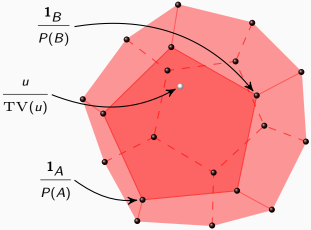

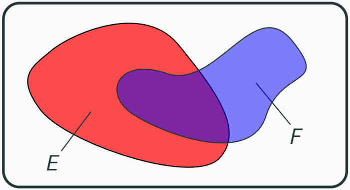

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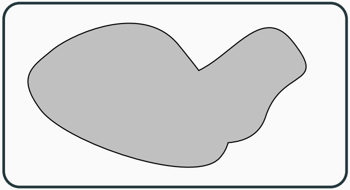

$\{u \geq 1\} = E \cup F$

Faces and chains

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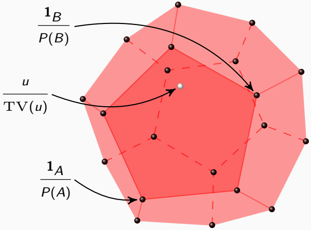
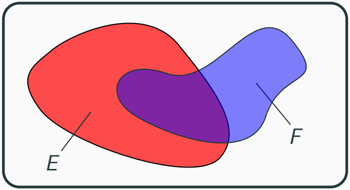


Chain

Increasing collection of sets whose indicator function belongs to $\mathcal{F}$



$u = 2\,\mathbf{1}_E + \mathbf{1}_F$

$\{u \geq 0\} = \mathbb{R}^2$

Support recovery (general case)

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Non-degenerate source condition

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- chains associated to u_0 are maximal in the face exposed by η_0

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Theorem

Assume that

- u_0 satisfies the non-degenerate source condition
- Φ^* is continuous from $\mathcal{H}$ to $\mathrm{C}^1_b(\mathbb{R}^2)$

then provided $\lambda \leq \lambda_0$ and $\|w\|_{\mathcal{H}}/\lambda \leq \alpha$, $u_{\lambda,w}$ has the same "sparsity" as u_0

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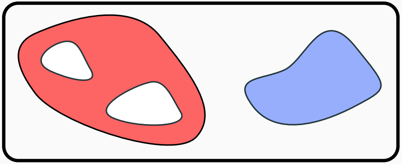
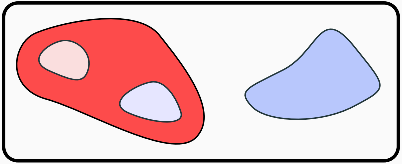


u_0



$u_{\lambda,w}$

Numerical resolution: a grid-free conditional gradient approach

Numerical resolution of $(\mathcal{P}_\lambda(y))$

Solve

$$\min_{u \in \mathrm{L}^2(\mathbb{R}^2)} \frac{1}{2}\|\Phi u - y\|^2 + \lambda\,\mathrm{TV}(u) \qquad (\mathcal{P}_\lambda(y))$$

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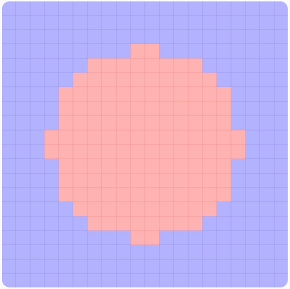
Fixed grid approximation

$$u = \sum_i \sum_j u_{ij}\,\mathbf{1}_{C_{ij}}$$



Discretizations of the total variation (images: [Tabti et al., 2018])

Anisotropic

- $\sum_{ij} |(D_x u)_{ij}| + |(D_y u)_{ij}|$
- Sharp edges, **grid bias**

Isotropic

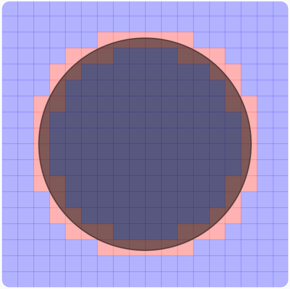
- $\sum_{ij} \sqrt{(D_x u)_{ij}^2 + (D_y u)_{ij}^2}$
- **Blur**

State of the art review: [Chambolle and Pock, 2021]

Numerical representation of simple images

Fixed grid

- $\mathcal{O}\left(1/h^2\right)$ pixels
- $\mathcal{O}\left(1/h\right)$ "relevant" pixels
- $u \mapsto \mathrm{TV}(u)$ convex



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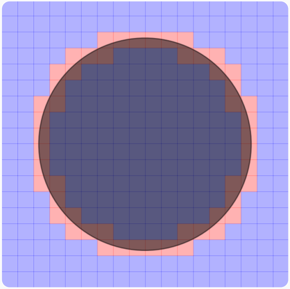
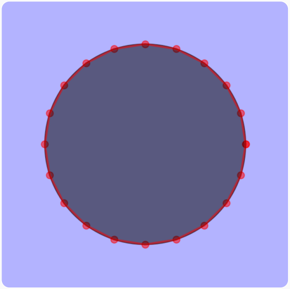
Boundary discretization

- More compact for simple img.
- Numerically more involved
- $E \mapsto \mathrm{TV}(\mathbf{1}_E)$ "non convex"





Frank-Wolfe based algorithm

Algorithm

[Bredies and Pikkarainen, 2013]
[Boyd et al., 2017, Denoyelle et al., 2019]

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$$\text{s.t.} \quad |E| < +\infty, \; 0 < P(E) < +\infty$$

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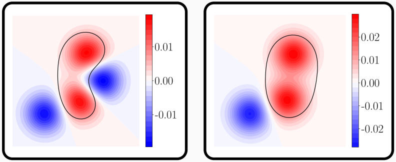

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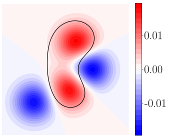

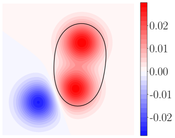

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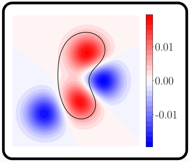

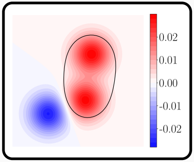

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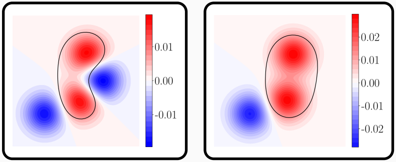

Iterates are linear combinations of indicator functions of simple sets

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- $u_{k+1} = \alpha_k\, u_k + \beta_k\, \mathbf{1}_{E_{k+1}}$
- Loc. opt. $(a, E) \mapsto F\left(\sum_i a_i \mathbf{1}_{E_i}\right)$

[Bredies and Pikkarainen, 2013]
[Boyd et al., 2017, Denoyelle et al., 2019]

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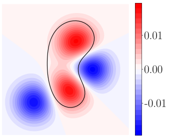
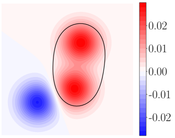




Iterates are linear combinations of indicator functions of simple sets

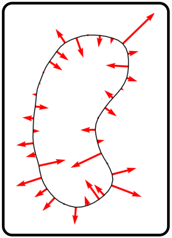
Two-step approximation of generalized Cheeger sets

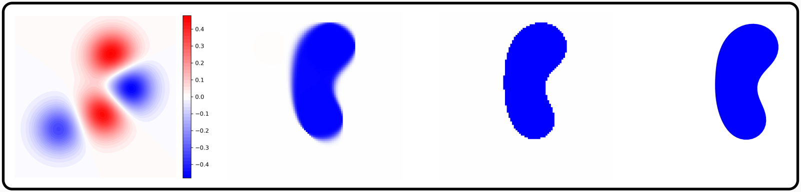
Fixed grid initialization [Carlier et al., 2009]

$$\text{Solve} \min_{u \in E^h} \langle \eta^h, u \rangle \text{ s.t. } \mathrm{TV}^h(u) \leq 1$$

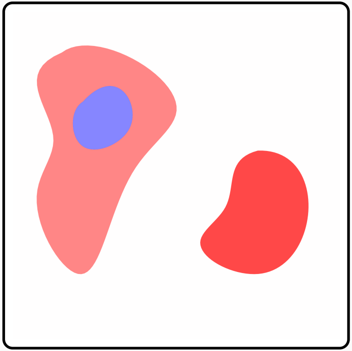
Shape gradient algorithm

- $\theta_n \in \operatorname{Argmax}_{\theta \in \Theta} \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \left[J\left((Id + \epsilon\, \theta)(E_n) \right) - J\left(E_n \right) \right]$
- $E_{n+1} = (Id + \epsilon_n\, \theta_n)(E_n)$

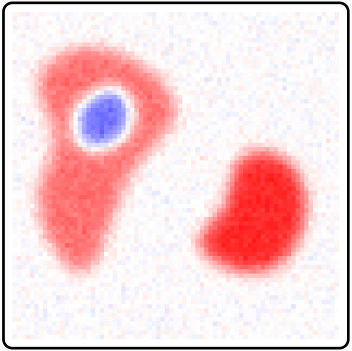




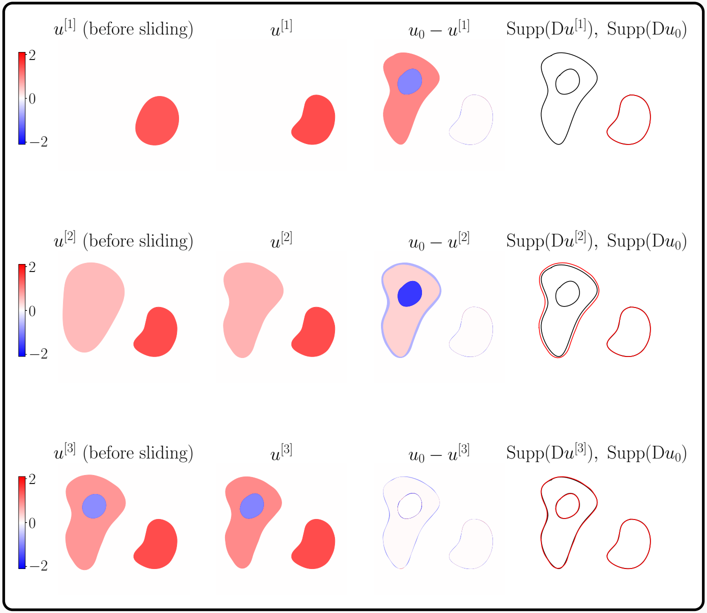
Implementation: `github.com/rpetit/PyCheeger`

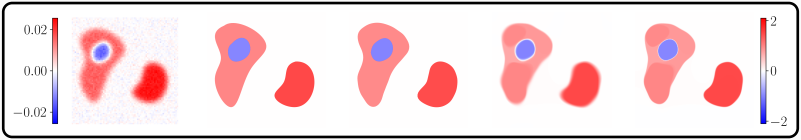


Unknown image u_0

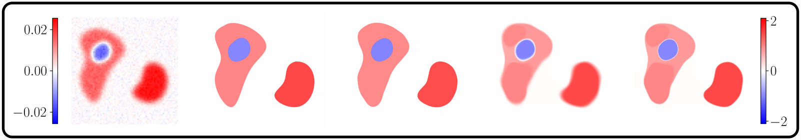


Observations $y = \Phi u_0 + w$


$u^{[1]}$ (before sliding)
$u^{[1]}$
$u_0 - u^{[1]}$
$\mathrm{Supp}(\mathrm{D}u^{[1]}),\ \mathrm{Supp}(\mathrm{D}u_0)$
2
0
−2
$u^{[2]}$ (before sliding)
$u^{[2]}$
$u_0 - u^{[2]}$
$\mathrm{Supp}(\mathrm{D}u^{[2]}),\ \mathrm{Supp}(\mathrm{D}u_0)$
2
0
−2
$u^{[3]}$ (before sliding)
$u^{[3]}$
$u_0 - u^{[3]}$
$\mathrm{Supp}(\mathrm{D}u^{[3]}),\ \mathrm{Supp}(\mathrm{D}u_0)$
2
0
−2



Left to right: observations, signal, ours, isotropic TV, “Condat’s” TV



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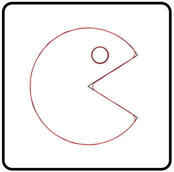
Signal u_0

Isotropic TV

Ours ($\hat{u}_{\lambda,w}$)

Observations

"Condat's" TV



$\mathrm{Supp}(\mathrm{D}u_0)$, $\mathrm{Supp}(\mathrm{D}\hat{u}_{\lambda,w})$

Perspectives

Recovery guarantees

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- Error bounds

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Numerical resolution

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Numerical resolution

- Finite time convergence?

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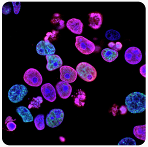
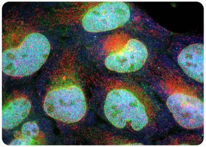
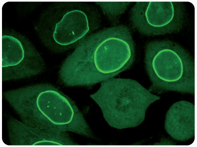
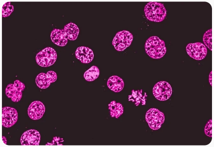

Perspectives

Recovery guarantees

- Error bounds
- Pre-certificates
- Sufficient identifiability conditions

Numerical resolution

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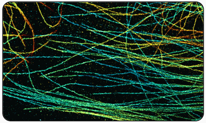
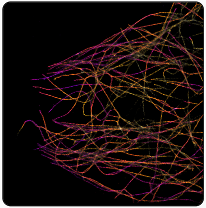
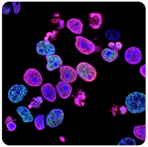
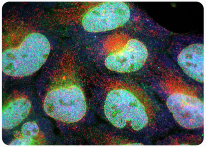
- Robust sliding step (topology changes)

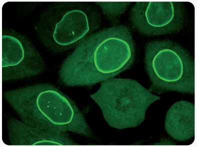
Recovery of 1D structures?

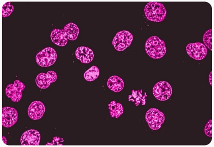












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