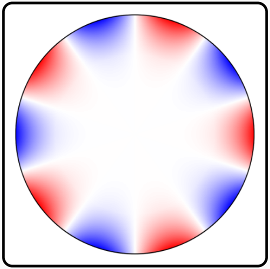
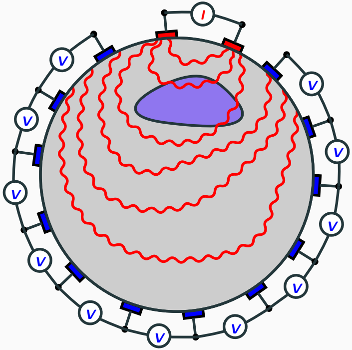


On the non-convexity issue in electrical impedance tomography

Romain Petit
joint work with Giovanni S. Alberti (University of Genoa)
and Clarice Poon (University of Warwick)

Electrical impedance tomography (EIT)



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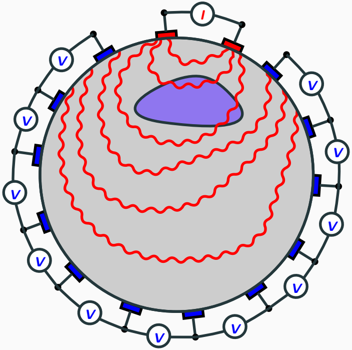

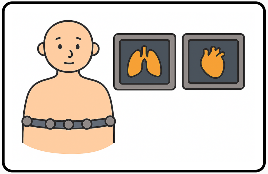

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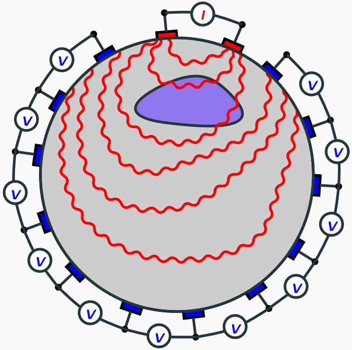

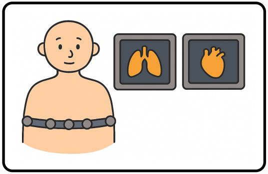

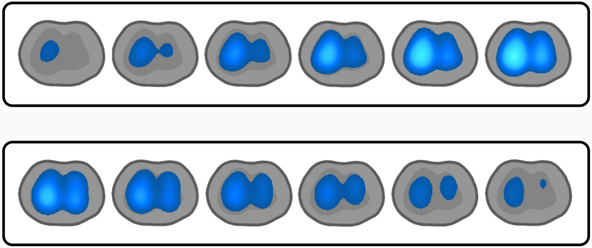

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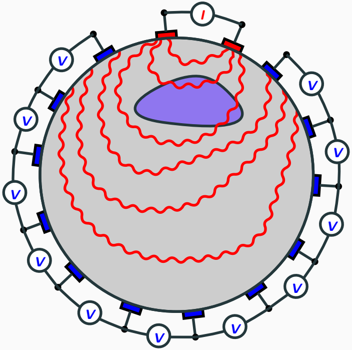

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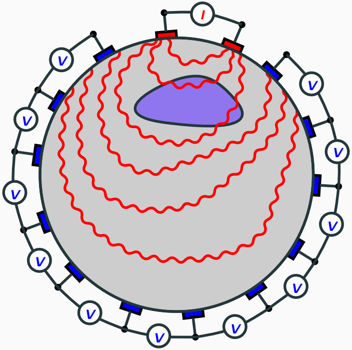

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- Introduced in [Calderón, 1980]

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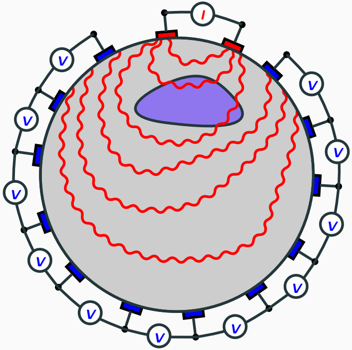

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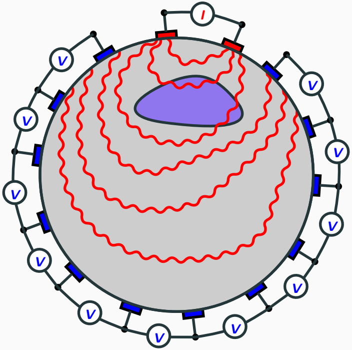

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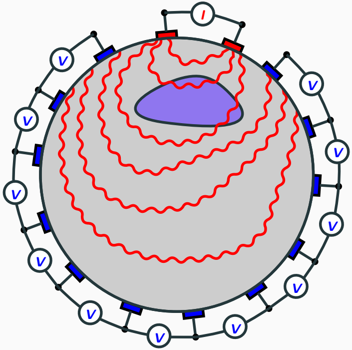

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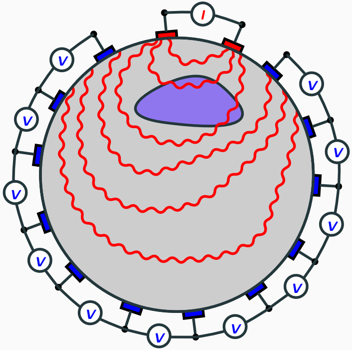

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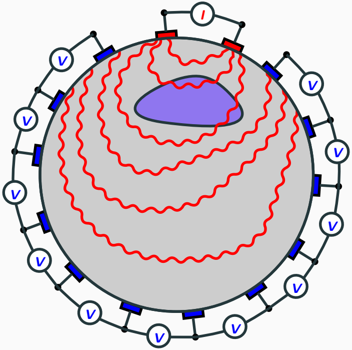

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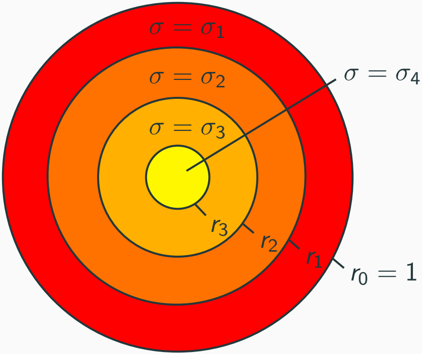
This talk

Investigate landscape of $\sigma \mapsto \|\Lambda(\sigma) - y\|_2^2$ in **radial setting**

[Sylvester and Uhlmann, 1987, Alessandrini, 1988, Alessandrini and Vessella, 2005]

Radial setting



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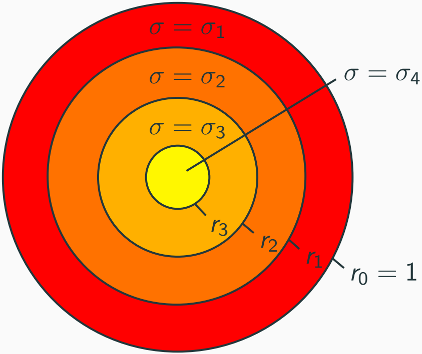
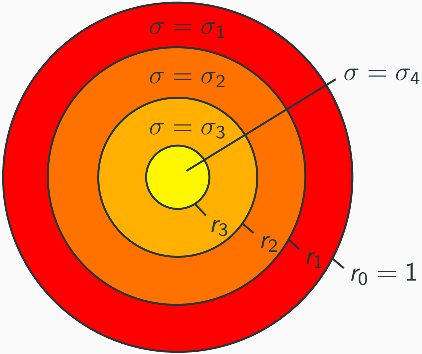


Forward map [Siltanen et al., 2000]

- If $g_j(\theta) = \cos(j\theta)$ then $\sigma \partial_\nu u_{|\partial\Omega} = \lambda_j(\sigma) g_j$.

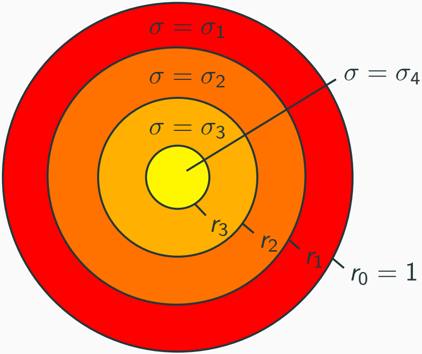
Radial setting



Forward map [Siltanen et al., 2000]

- If $g_j(\theta) = \cos(j\theta)$ then $\sigma \partial_\nu u_{|\partial\Omega} = \lambda_j(\sigma) g_j$.
- Define $\Lambda : \sigma \in \mathbb{R}^n \mapsto [\lambda_j(\sigma)]_{1 \leq j \leq m} \in \mathbb{R}^m$.

Radial setting



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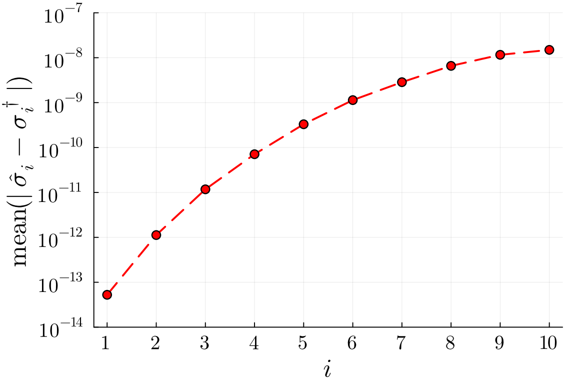
- Define $\Lambda : \sigma \in \mathbb{R}^n \mapsto [\lambda_j(\sigma)]_{1 \leq j \leq m} \in \mathbb{R}^m$.
- Evaluate Λ by solving linear system (+ derivatives via autodiff)

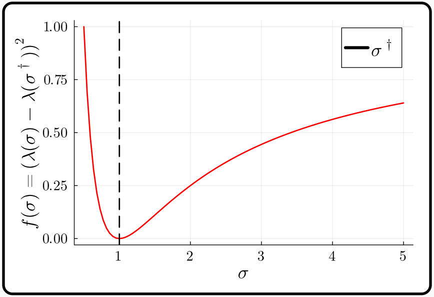
Ill-posedness

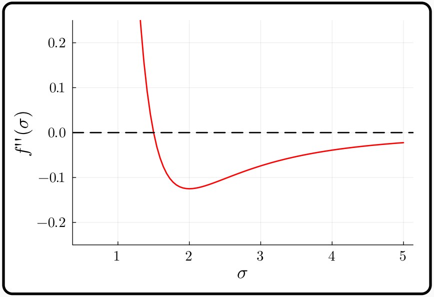
Setting

- Draw $\hat{\sigma}, \sigma^\dagger$ such that $\|\Lambda(\hat{\sigma}) - \Lambda(\sigma^\dagger)\|_\infty \leq 10^{-15}$.
- Compute mean error $|\hat{\sigma}_i - \sigma_i^\dagger|$ on each annulus



One-dimensional case





Identifiability and absence of bad critical points

Forward map $\Lambda : \sigma \mapsto (\lambda_j(\sigma))_{1 \leq j \leq m}$

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Conjecture

If $m \geq n$, the following holds.

- If $\Lambda(\sigma) = \Lambda(\sigma^\dagger)$ then $\sigma = \sigma^\dagger$.
- The unique critical point of $\sigma \mapsto \|\Lambda(\sigma) - \Lambda(\sigma^\dagger)\|_2^2$ is $\sigma = \sigma^\dagger$.

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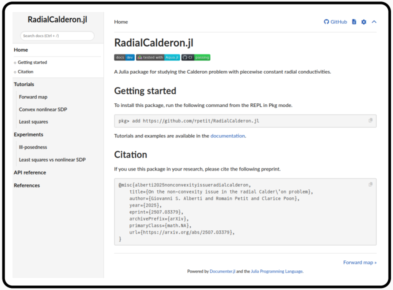
Partial results

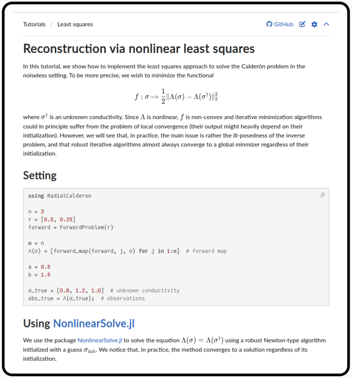
- Full proof for $n = 2$ (proof of case $m > 2$ by Irène Waldspurger).
- Partial proof for $n > 2$ under a numerically verifiable criterion.

Code: `rpetit.github.io/RadialCalderon.jl`

Content

- Implementation of forward map (+ derivatives via autodiff)
- Least squares approach
- Extensive benchmark against [Harrach, 2023] (convex programming)





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- Mild non-convexity.
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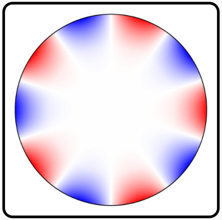

Preprint

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`arXiv:2507.03379`

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