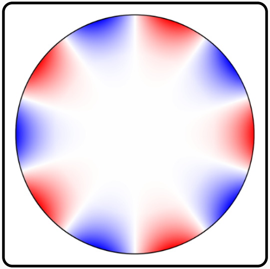

Nonlinear inverse problems: the example of electrical impedance tomography

Romain Petit

CIL winter school, February 25th

Linear versus non-linear inverse problems

Unknown signal x_0

Linear versus non-linear inverse problems

Unknown signal x_0

Observations $y = A(x_0)$

Linear versus non-linear inverse problems

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Inverse problem

Reconstruct x_0 from y

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Linear problems

- A linear
- Solving $A(x) = y$ easy

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Non-linear problems

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Linear versus non-linear inverse problems

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Linear problems

- A linear
- $x \mapsto \|A(x) - y\|^2$ convex

Non-linear problems

- A non-linear
- $x \mapsto \|A(x) - y\|^2$ non-convex

This talk

Examples of non-linear IPs

- Why do we care?
- Where do they appear?

This talk

Examples of non-linear IPs

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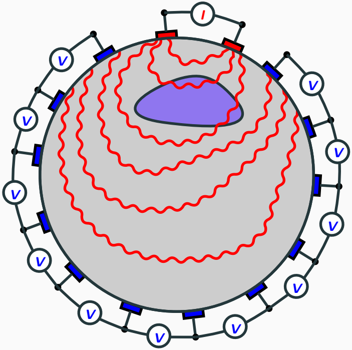
- Where do they appear?



Focus: electrical impedance tompography (EIT)

- Applications
- Theoretical and computational aspects
- How to solve $A(x) = y$
- Focus on radial case

Examples of non-linear inverse problems

Phase retrieval (image source: [Pinilla et al., 2018])

Inverse problem

- Unknown $x = (x_i)_{i=1}^n \in \mathbb{C}^n$
- Obs. $y = (y_k)_{k=1}^m$ with $y_k = |\langle w^k, x \rangle|$ and $w^k \in \mathbb{C}^n$

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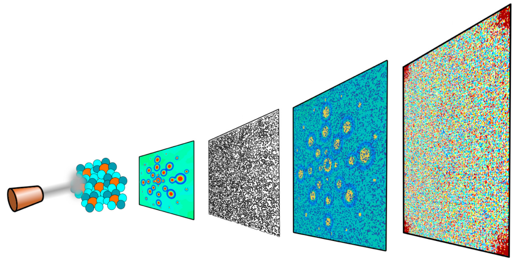

Applications

- Diffraction imaging
- X-ray cristallography

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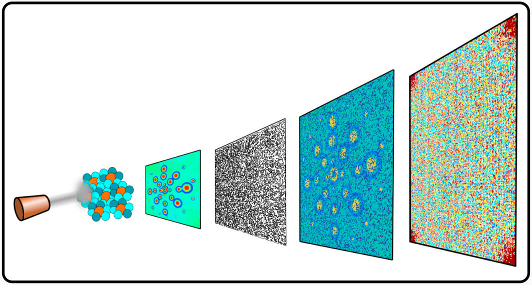

Applications

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- X-ray cristallography

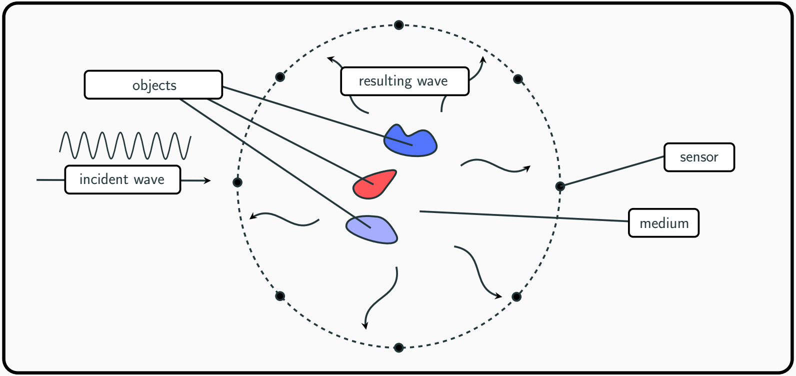
More details

- Phase retrieval with app. to optical im.: a contemporary overview (Schechtman et al.)
- Lecture notes on non-convex alg. for low-rank matrix recovery (Waldspurger)

Inverse problems for partial differential equations (PDEs)

Usual scenario

- Want to measure property of object
- Send wave → interact. with object → measure resulting wave
- Recover property from resulting wave → non-linear IP



Inverse problems for PDEs

Gravimetry

- Recover mass distrib. from gravitational field
- Recover f from u with $u'' = f \qquad (A : f \mapsto u)$
- Inverse **source** problem
- **Linear**: if $u_1'' = f_1$ and $u_2'' = f_2$ then $(u_1 + u_2)'' = (f_1 + f_2)$

Inverse problems for PDEs

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Scattering

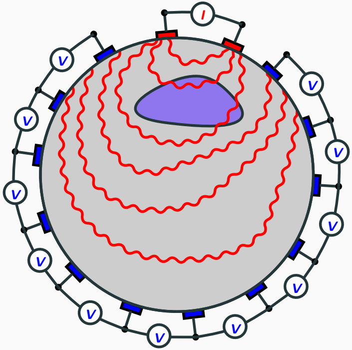
- Recover position of obstacle from scattered wave
- Recover k from u with $u'' + ku = 0 \qquad (A : k \mapsto u)$
- Inverse **coefficient** problem
- **Nonlinear**: if $u_1'' + k_1 u_1 = 0$ and $u_2'' + k_2 u_2 = 0$ then

$$(u_1 + u_2)'' + (k_1 + k_2)(u_1 + u_2) \neq 0$$

Electrical impedance tomography

Joint work with G. S. Alberti, C. Poon and I. Waldspurger

Electrical impedance tomography (EIT)



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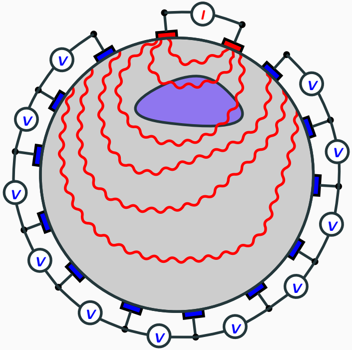

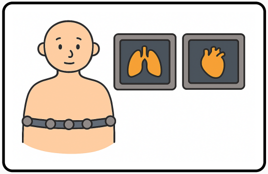

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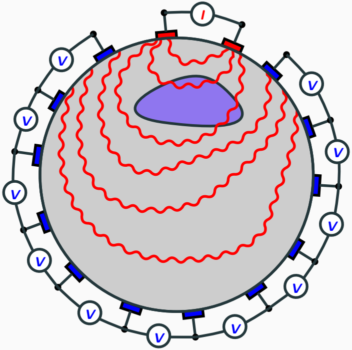

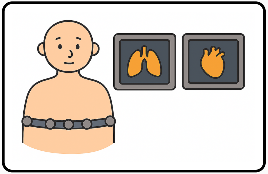

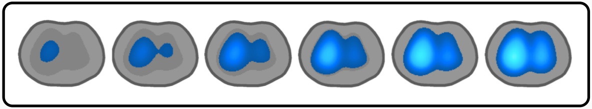

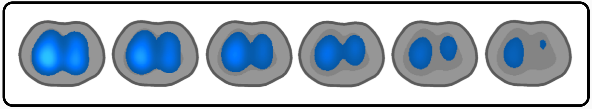

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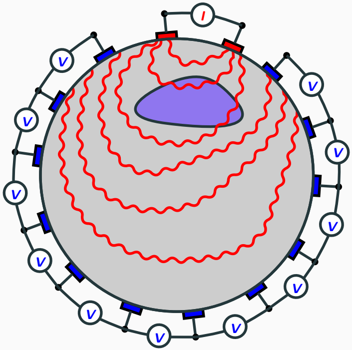

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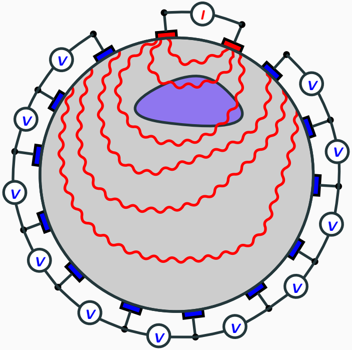

Model (Calderón problem)

- Introduced in [Calderón, 1980]

[Sylvester and Uhlmann, 1987, Alessandrini, 1988, Alessandrini and Vessella, 2005]

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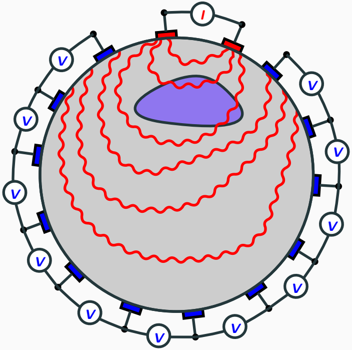

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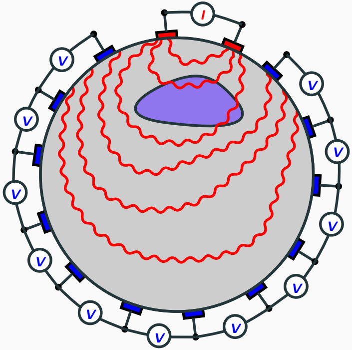

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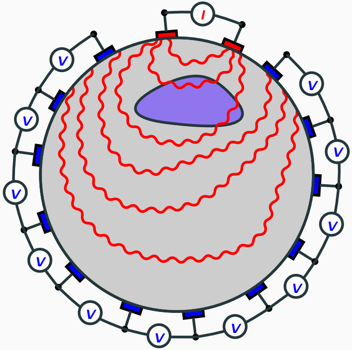

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- Ill-posed nonlinear inverse problem

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Theoretical properties of EIT

Identifiability

- Voltage-current map determines σ
  [Kohn and Vogelius, 1984, Sylvester and Uhlmann, 1987]
- Finitely many measurements [Alberti and Santacesaria, 2019]

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Ill-posedness

- At best logarithmic stability
  [Alessandrini, 1988, Alessandrini and Vessella, 2005]
- Worsens with distance to boundary
  [Garde and Knudsen, 2017, Alessandrini and Scapin, 2017]

Computational aspects of EIT

Forward map

- Evaluation $\to$ solve PDE for each boundary potential
- FEM, surrogate models, etc.
- More faithful models (complete electrode model)

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Notable software libraries

- pyEIT (2018): FEM, linearization or least squares
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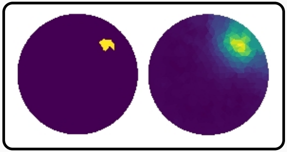
Main problems

- Poor spatial resolution
- Ill-posedness $\nearrow$ with dist. to bound.
- Regularization difficult



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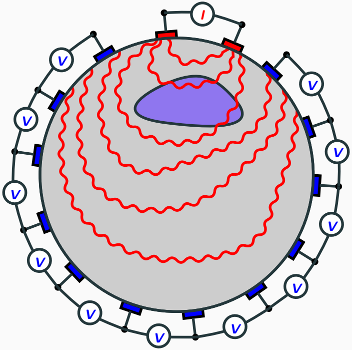

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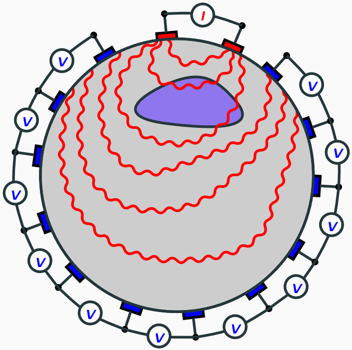

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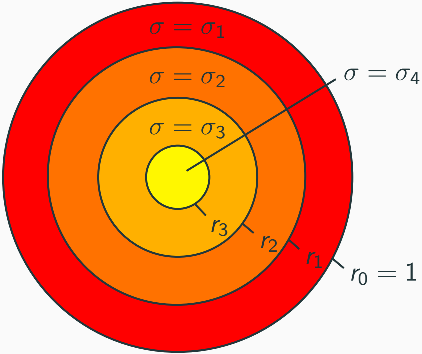
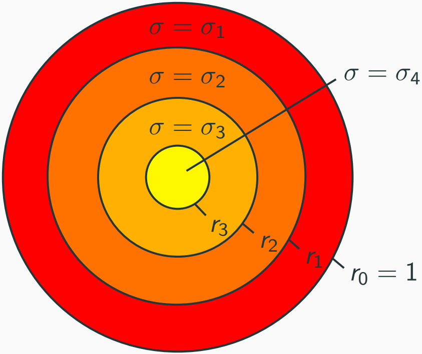
This talk

Investigate landscape
of $\sigma \mapsto \|\Lambda(\sigma) - y\|_2^2$
in **radial setting**

Radial setting



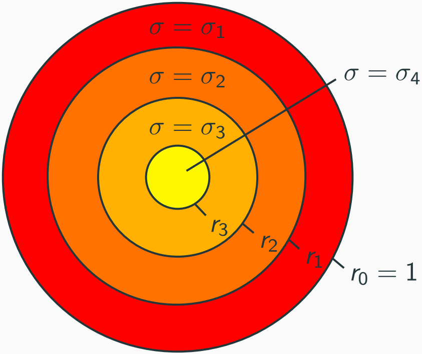
Radial setting



Forward map [Siltanen et al., 2000]

- If boundary potential $g_j(\theta) = \cos(j\theta)$ then boundary current $= \lambda_j(\sigma) g_j$.

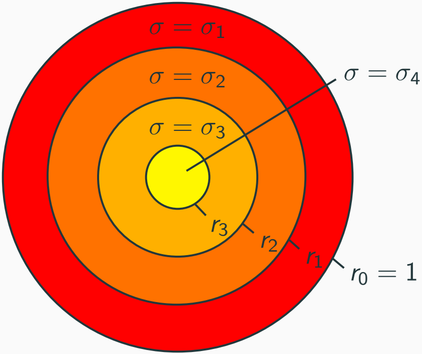
Radial setting



Forward map [Siltanen et al., 2000]

- If boundary potential $g_j(\theta) = \cos(j\theta)$ then boundary current $= \lambda_j(\sigma) g_j$.
- Define $\Lambda : \sigma \in \mathbb{R}^n \mapsto [\lambda_j(\sigma)]_{1 \leq j \leq m} \in \mathbb{R}^m$.

Radial setting



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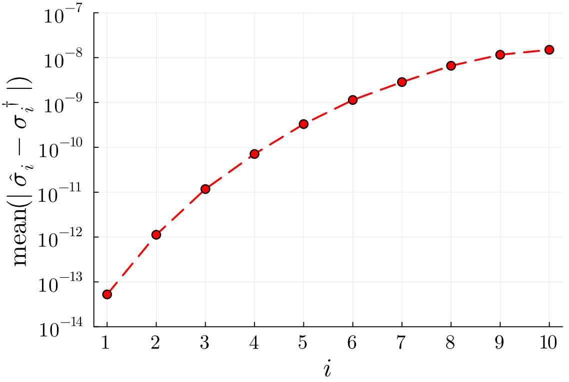
- Define $\Lambda : \sigma \in \mathbb{R}^n \mapsto [\lambda_j(\sigma)]_{1 \leq j \leq m} \in \mathbb{R}^m$.
- Evaluate Λ by solving linear system (+ derivatives via autodiff)

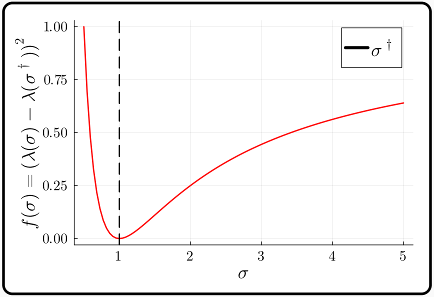
Ill-posedness

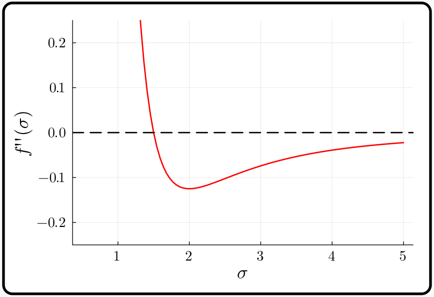
Setting

- Draw $\hat{\sigma}, \sigma^\dagger$ such that $\|\Lambda(\hat{\sigma}) - \Lambda(\sigma^\dagger)\|_\infty \leq 10^{-15}$.
- Compute mean error $|\hat{\sigma}_i - \sigma_i^\dagger|$ on each annulus



One-dimensional case





Two-dimensional case

2D plots

Identifiability and absence of bad critical points

Forward map $\Lambda : \sigma \mapsto (\lambda_j(\sigma))_{1 \leq j \leq m}$

Identifiability and absence of bad critical points

Forward map $\Lambda : \sigma \mapsto (\lambda_j(\sigma))_{1 \leq j \leq m}$

Conjecture

If $m \geq n$, the following holds.

- If $\Lambda(\sigma) = \Lambda(\sigma^\dagger)$ then $\sigma = \sigma^\dagger$.
- The unique critical point of $\sigma \mapsto \|\Lambda(\sigma) - \Lambda(\sigma^\dagger)\|_2^2$ is $\sigma = \sigma^\dagger$.

Identifiability and absence of bad critical points

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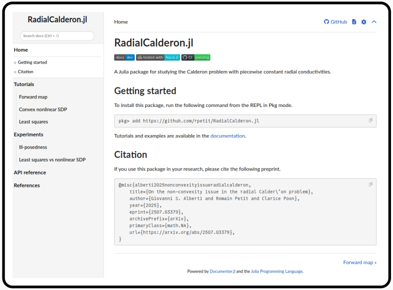
Partial results (Alberti, P., Poon, Waldspurger '26)

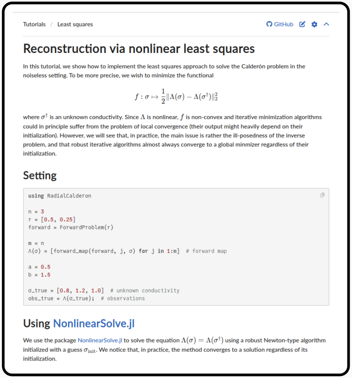
- Full proof for $n = 2$.
- Partial proof for $m = n > 2$ under a numerically verifiable criterion.
- Extensive empirical evidence that least squares solvers find $\sigma^\dagger$.

Code: rpetit.github.io/RadialCalderon.jl

Content

- Implementation of forward map (+ derivatives via autodiff)
- Least squares approach
- Extensive benchmark against [Harrach, 2023] (convex programming)





Conclusion on EIT

Summary

- Mild non-convexity.
- True problem: ill-posedness (+ interaction with nonlinearity).

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- Tailored algorithms with better performance?
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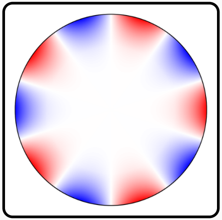

Article

On the non-convexity issue in the radial Calderón problem

`SIAM J. on Imaging Sciences`

Summary

Non-linear inverse problems

- Ubiquitous in imaging sciences
- Complex physics (wave interaction) $\rightarrow$ PDE $\rightarrow$ non-linear IP
- Harder than linear IP

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- Complex physics (wave interaction) $\rightarrow$ PDE $\rightarrow$ non-linear IP
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- Even when non-convexity mild regularization difficult topic

Perspective: learning-based approaches

- Big distinction between linear and non-linear?
- Physics-informed models
- Incorporate a priori info.

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Details

Forward map [Siltanen et al., 2000]

$$\lambda_j(\sigma) = \frac{1 + C_{1,j}}{j\sigma_1(1 - C_{1,j})}$$

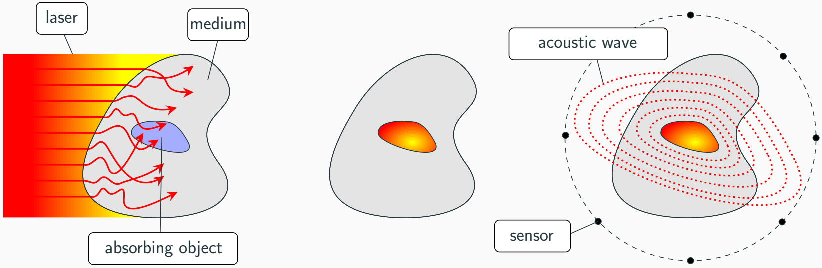
$$C_{n,j} = 0 \quad \text{and} \quad C_{i,j} = \frac{\rho_i r_i^{2j} + C_{i+1,j}}{1 + \rho_i C_{i+1,j} r_i^{-2j}}$$

$$\rho_i = \frac{\sigma_i - \sigma_{i+1}}{\sigma_i + \sigma_{i+1}}$$

2D case

$$\lambda_j(\sigma) = \frac{1 + \rho_1 r_1^{2j}}{j\sigma_1(1 - \rho_1 r_1^{2j})}$$

Photo-acoustic tomography

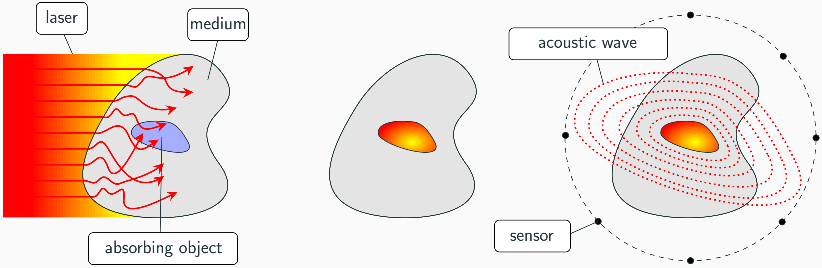


Optical absorption

Raise of temp. & pressure

Propagation and detection

Photo-acoustic tomography



Optical absorption   Raise of temp. & pressure   Propagation and detection

Simplified model (2nd step) (see e.g. [Alberti and Capdeboscq, 2018])

- $-\Delta u + qu = 0$ in Ω, $u = g$ on $\partial\Omega$
- Reconstruct q in Ω from u in Ω and q on $\partial\Omega$