

Nonlinear inverse problems: the example of electrical impedance tomography

Romain Petit

CIL winter school, February 25th

Linear versus non-linear inverse problems



Unknown signal x_0

Linear versus non-linear inverse problems



Unknown signal x_0



Observations $y = A(x_0)$

Linear versus non-linear inverse problems



Unknown signal x_0



Observations $y = A(x_0)$

Inverse problem

Reconstruct x_0 from y

Linear versus non-linear inverse problems



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Linear problems

- A linear
- Solving $A(x) = y$ easy

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Non-linear problems

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Linear problems

- A linear
- $x \mapsto \|A(x) - y\|^2$ convex

Non-linear problems

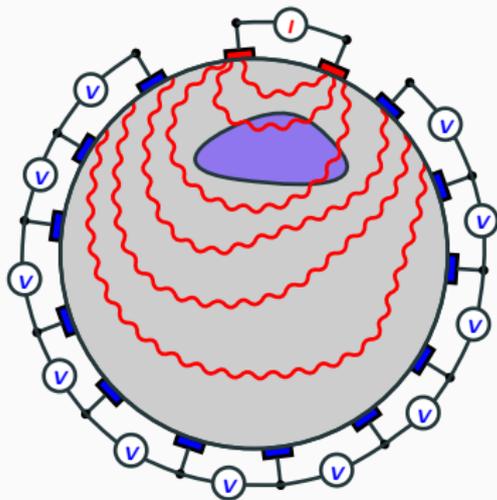
- A non-linear
- $x \mapsto \|A(x) - y\|^2$ non-convex

Examples of non-linear IPs

- Why do we care?
- Where do they appear?

Examples of non-linear IPs

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- Where do they appear?



Focus: electrical impedance tomography (EIT)

- Applications
- Theoretical and computational aspects
- How to solve $A(x) = y$
- Focus on radial case

Examples of non-linear inverse problems

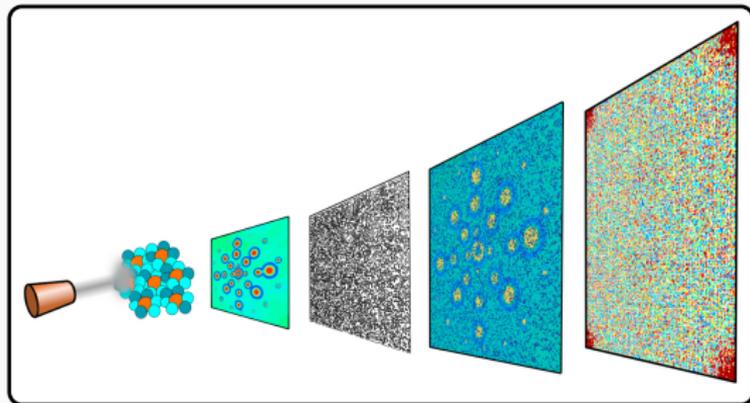
Inverse problem

- Unknown $x = (x_i)_{i=1}^n \in \mathbb{C}^n$
- Obs. $y = (y_k)_{k=1}^m$ with $y_k = |\langle w^k, x \rangle|$ and $w^k \in \mathbb{C}^n$

Phase retrieval (image source: [Pinilla et al., 2018])

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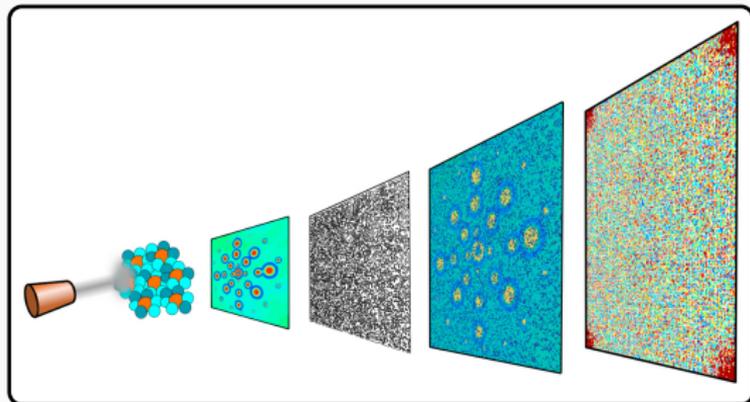
Applications

- Diffraction imaging
- X-ray crystallography

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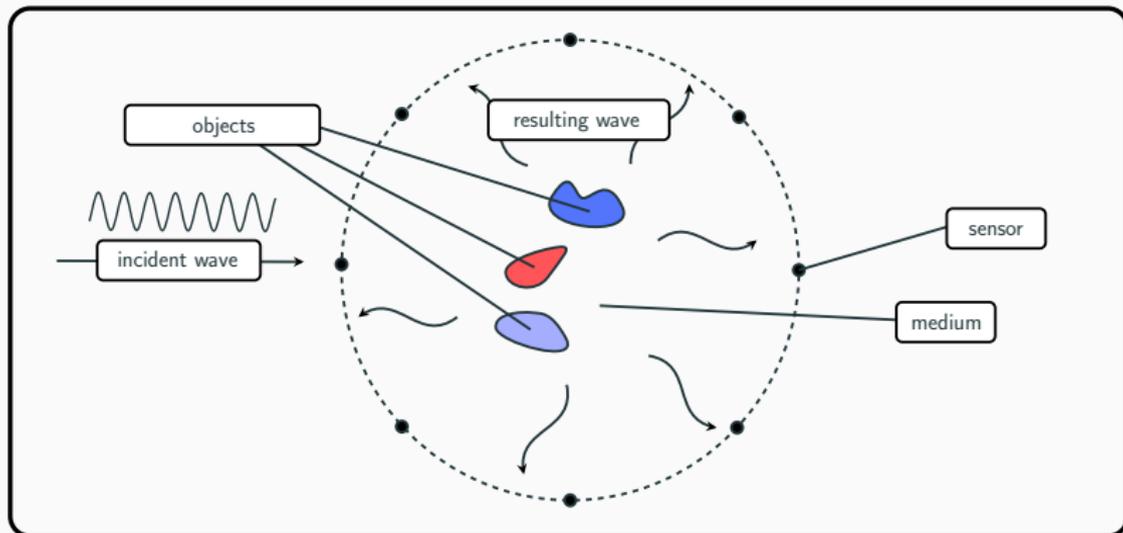
More details

- Phase retrieval with app. to optical im.: a contemporary overview (Schechtman et al.)
- Lecture notes on non-convex alg. for low-rank matrix recovery (Waldspurger)

Inverse problems for partial differential equations (PDEs)

Usual scenario

- Want to measure property of object
- Send wave \rightarrow interact. with object \rightarrow measure resulting wave
- Recover property from resulting wave \rightarrow non-linear IP



Gravimetry

- Recover mass distrib. from gravitational field
- Recover f from u with $u'' = f$ ($A : f \mapsto u$)
- Inverse **source** problem
- **Linear**: if $u_1'' = f_1$ and $u_2'' = f_2$ then $(u_1 + u_2)'' = (f_1 + f_2)$

Inverse problems for PDEs

Gravimetry

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Scattering

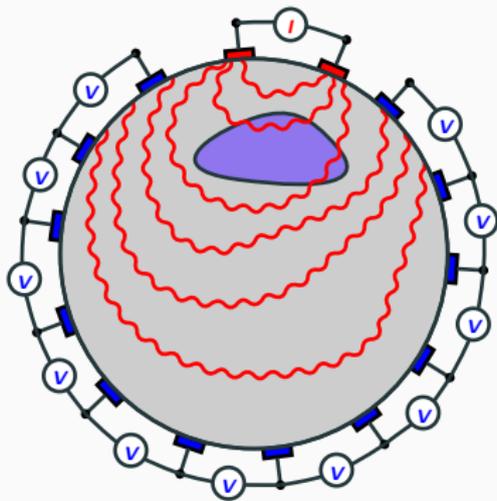
- Recover position of obstacle from scattered wave
- Recover k from u with $u'' + ku = 0$ ($A : k \mapsto u$)
- Inverse **coefficient** problem
- **Nonlinear**: if $u_1'' + k_1 u_1 = 0$ and $u_2'' + k_2 u_2 = 0$ then

$$(u_1 + u_2)'' + (k_1 + k_2)(u_1 + u_2) \neq 0$$

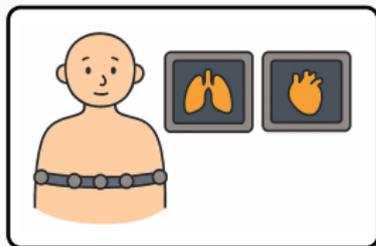
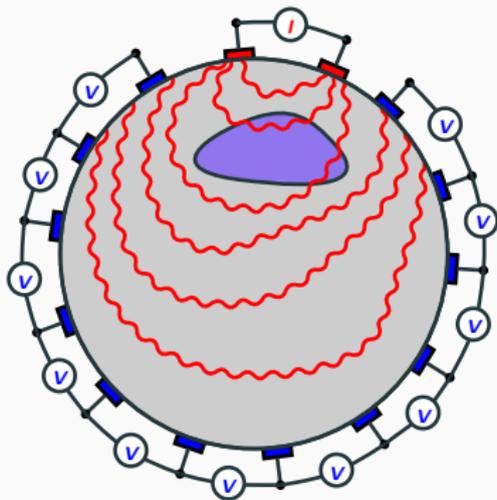
Electrical impedance tomography

Joint work with G. S. Alberti, C. Poon and I. Waldspurger

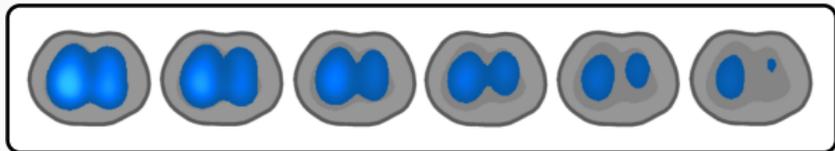
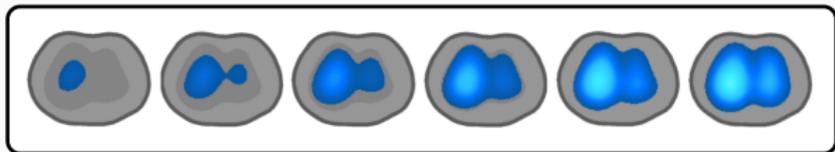
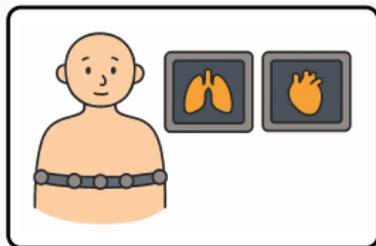
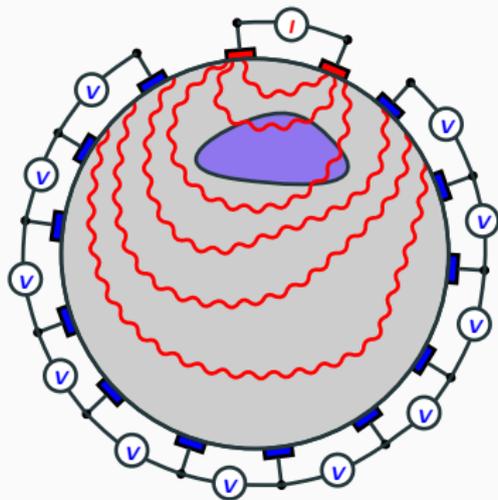
Electrical impedance tomography (EIT)



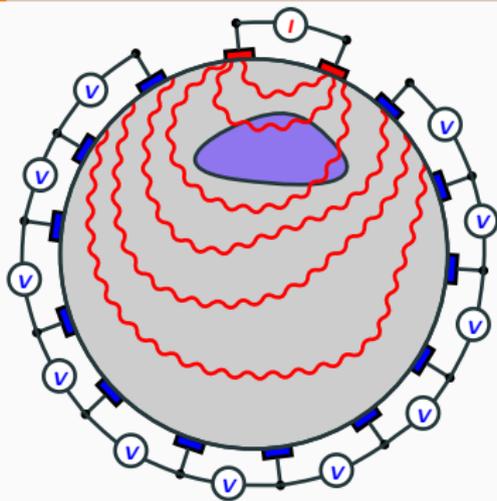
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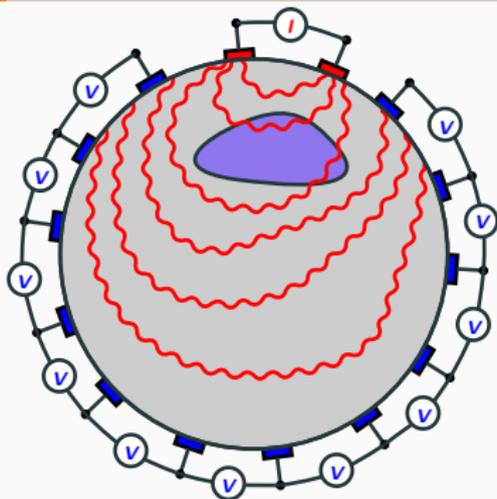
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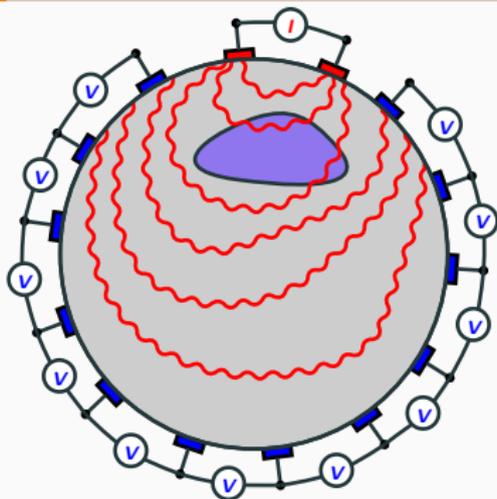
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Model (Calderón problem)

- Introduced in [Calderón, 1980]

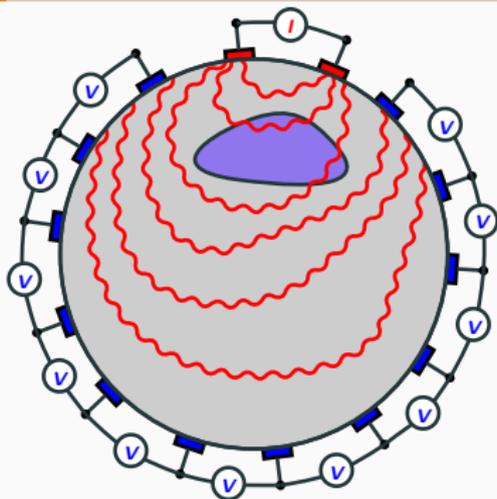
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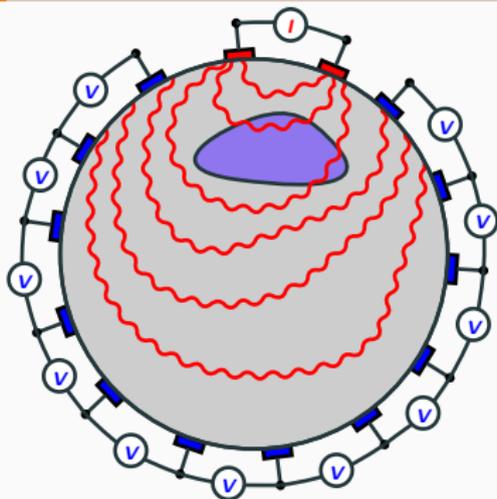
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- Ill-posed nonlinear inverse problem

Identifiability

- Voltage-current map determines σ
[Kohn and Vogelius, 1984, Sylvester and Uhlmann, 1987]
- Finitely many measurements [Alberti and Santacesaria, 2019]

Theoretical properties of EIT

Identifiability

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Ill-posedness

- At best logarithmic stability
[Alessandrini, 1988, Alessandrini and Vessella, 2005]
- Worsens with distance to boundary
[Garde and Knudsen, 2017, Alessandrini and Scapin, 2017]

Forward map

- Evaluation \rightarrow solve PDE for each boundary potential
- FEM, surrogate models, etc.
- More faithful models (complete electrode model)

Computational aspects of EIT

Forward map

- Evaluation → solve PDE for each boundary potential
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Notable software libraries

- pyEIT (2018): FEM, linearization or least squares
- OpenEIT (2019, Mindseye Labs): dashboard and kit for real-time EIT

Computational aspects of EIT

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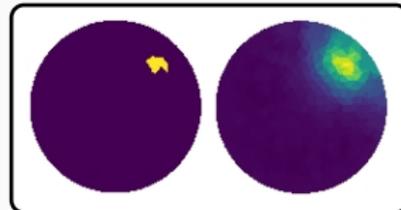
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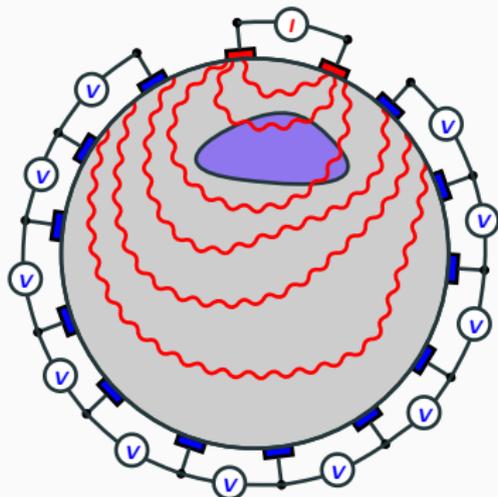
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Main problems

- Poor spatial resolution
- Ill-posedness \nearrow with dist. to bound.
- Regularization difficult



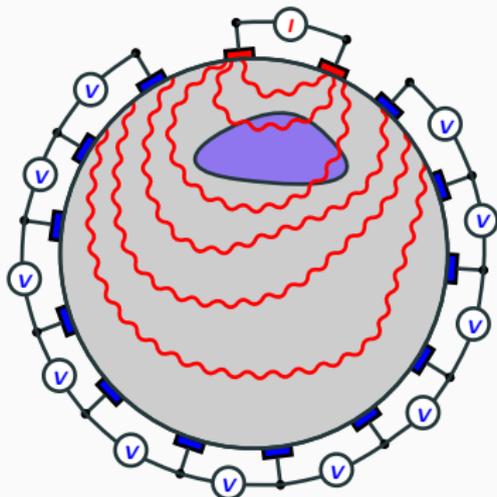
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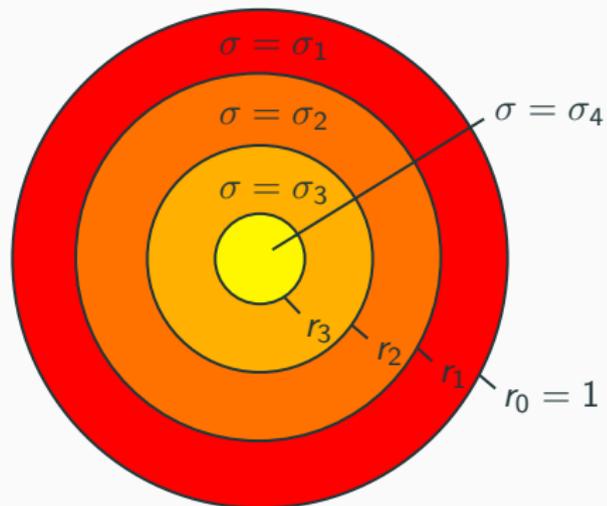
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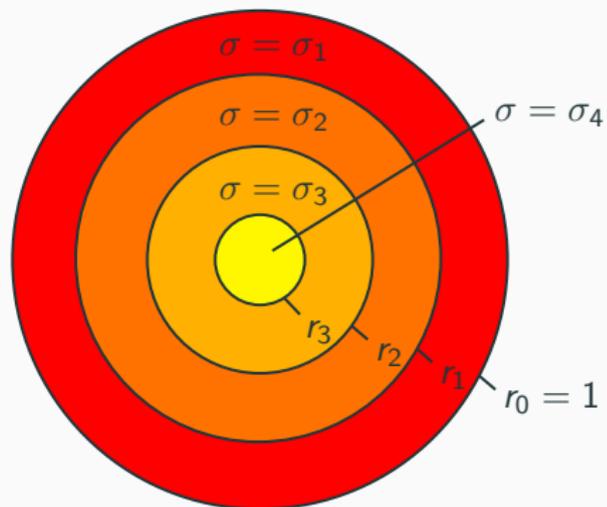
This talk

Investigate landscape
of $\sigma \mapsto \|\Lambda(\sigma) - y\|_2^2$
in **radial setting**

Radial setting



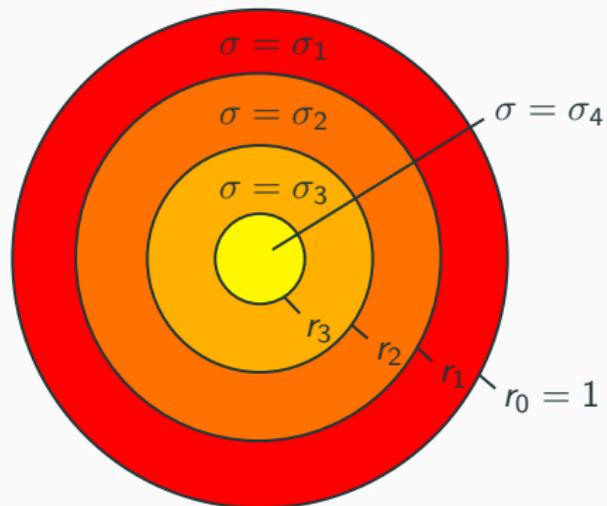
Radial setting



Forward map [Siltanen et al., 2000]

- If boundary potential $g_j(\theta) = \cos(j\theta)$ then boundary current $= \lambda_j(\sigma)g_j$.

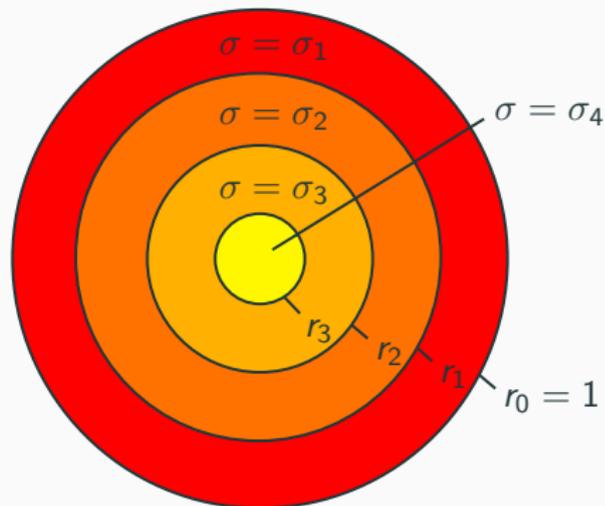
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- Define $\Lambda : \sigma \in \mathbb{R}^n \mapsto [\lambda_j(\sigma)]_{1 \leq j \leq m} \in \mathbb{R}^m$.

Radial setting

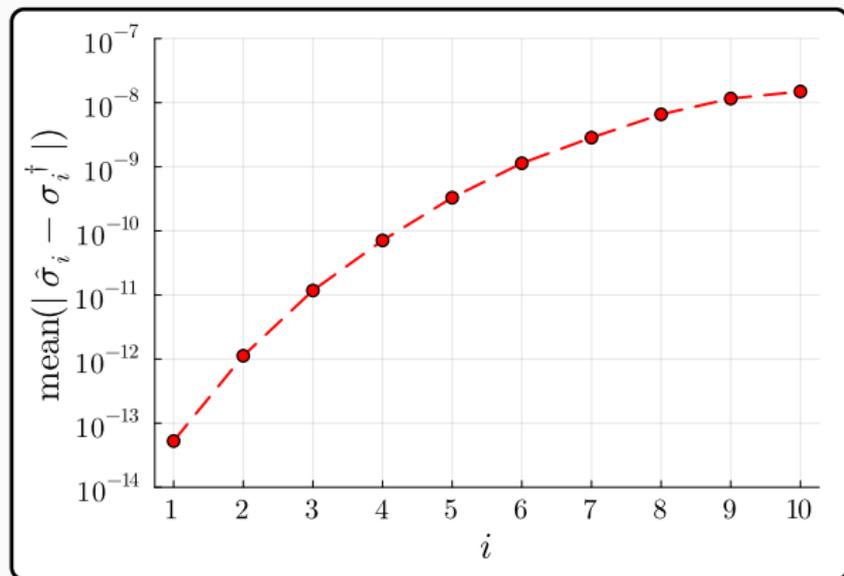


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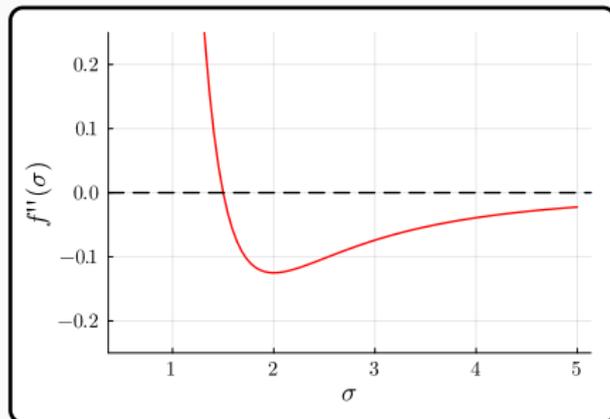
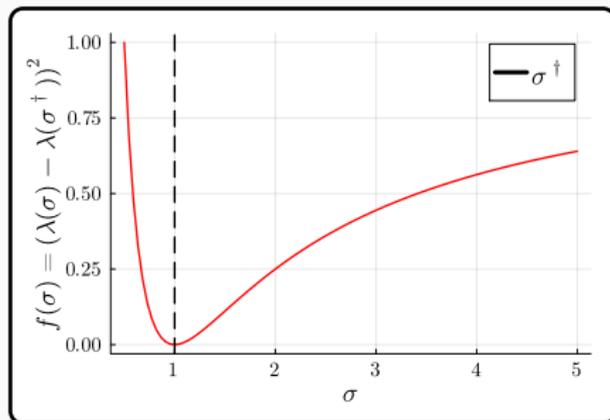
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- Define $\Lambda : \sigma \in \mathbb{R}^n \mapsto [\lambda_j(\sigma)]_{1 \leq j \leq m} \in \mathbb{R}^m$.
- Evaluate Λ by solving linear system (+ derivatives via autodiff)

Setting

- Draw $\hat{\sigma}, \sigma^\dagger$ such that $\|\Lambda(\hat{\sigma}) - \Lambda(\sigma^\dagger)\|_\infty \leq 10^{-15}$.
- Compute mean error $|\hat{\sigma}_i - \sigma_i^\dagger|$ on each annulus



One-dimensional case



2D plots

Identifiability and absence of bad critical points

Forward map $\Lambda : \sigma \mapsto (\lambda_j(\sigma))_{1 \leq j \leq m}$

Identifiability and absence of bad critical points

Forward map $\Lambda : \sigma \mapsto (\lambda_j(\sigma))_{1 \leq j \leq m}$

Conjecture

If $m \geq n$, the following holds.

- If $\Lambda(\sigma) = \Lambda(\sigma^\dagger)$ then $\sigma = \sigma^\dagger$.
- The unique critical point of $\sigma \mapsto \|\Lambda(\sigma) - \Lambda(\sigma^\dagger)\|_2^2$ is $\sigma = \sigma^\dagger$.

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Partial results (Alberti, P., Poon, Waldspurger '26)

- Full proof for $n = 2$.
- Partial proof for $m = n > 2$ under a numerically verifiable criterion.
- Extensive empirical evidence that least squares solvers find σ^\dagger .

Content

- Implementation of forward map (+ derivatives via autodiff)
- Least squares approach
- Extensive benchmark against [Harrach, 2023] (convex programming)

The screenshot shows the GitHub repository page for RadialCalderon.jl. The page includes a search bar, navigation links (Home, Getting started, Clusters, Tutorials, Forward map, Convex nonlinear ICP, Least squares, Experiments, IP problems, Least squares in nonlinear ICP, API reference, References), and a main content area. The main content area features a 'Getting started' section with instructions on how to install the package using the REPL in a Jupyter notebook. A 'Citation' section provides a BibTeX entry for the package. The page is powered by Documenter.jl and the Julia Programming Language.

The screenshot shows the 'Reconstruction via nonlinear least squares' tutorial page. The page includes a navigation bar with 'Tutorials / Least squares' and GitHub links. The main content area features a section titled 'Reconstruction via nonlinear least squares' with an introduction to the problem and a mathematical formulation of the forward map. The forward map is defined as $f: \sigma \mapsto \frac{1}{2} [\Lambda(\sigma) - \Lambda(\sigma^1)]^2$, where σ^1 is an unknown conductivity. The text explains that since Λ is nonlinear, f is non-convex and iterative minimization algorithms could in principle suffer from the problem of local convergence. However, the main issue is rather the ill-posedness of the inverse problem, and that robust iterative algorithms almost always converge to a global minimizer regardless of their initialization. A 'Setting' section shows a code snippet for setting up the problem. The code defines $n=3$, $r=[0.5, 0.25]$, and $\text{forward} = \text{ForwardProblem}(r)$. It then defines $\Lambda(\sigma) = [\text{forward_map}(\text{forward}, j, \sigma) \text{ for } j \text{ in } 1:n]$ and forward_map . The parameters are set to $a=0.5$ and $b=1.5$. The observations are defined as $\text{obs_true} = [0.8, 1.2, 1.0]$ and $\text{obs} = \text{observations}$. A 'Using NonlinearSolve.jl' section explains that the package NonlinearSolve.jl is used to solve the equation $\Lambda(\sigma) = \Lambda(\sigma^1)$ using a robust Newton-type algorithm initialized with a guess σ_{guess} . The text notes that, in practice, the method converges to a solution regardless of its initialization.

Summary

- Mild non-convexity.
- True problem: ill-posedness (+ interaction with nonlinearity).

Conclusion on EIT

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Perspectives

- Tailored algorithms with better performance?
- Study landscape in noisy regularized setting.

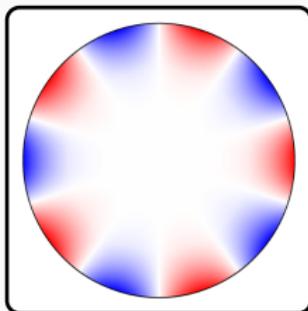
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Article

On the non-convexity issue in the
radial Calderón problem

SIAM J. on Imaging Sciences

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- Complex physics (wave interaction) \rightarrow PDE \rightarrow non-linear IP
- Harder than linear IP

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- Even when non-convexity mild regularization difficult topic

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Perspective: learning-based approaches

- Big distinction between linear and non-linear?
- Physics-informed models
- Incorporate a priori info.

-  Alberti, G. S. and Capdeboscq, Y. (2018).
Lectures on Elliptic Methods for Hybrid Inverse Problems, volume 25 of Cours Spécialisés (Specialized Courses).
Société Mathématique de France, Paris.
-  Alberti, G. S. and Santacesaria, M. (2019).
Calderón's inverse problem with a finite number of measurements.
Forum of Mathematics, Sigma, 7:e35.
-  Alessandrini, G. (1988).
Stable determination of conductivity by boundary measurements.
Applicable Analysis, 27(1-3):153–172.



Alessandrini, G. and Scapin, A. (2017).

Depth dependent resolution in Electrical Impedance Tomography.

Journal of Inverse and Ill-posed Problems, 25(3):391–402.



Alessandrini, G. and Vessella, S. (2005).

Lipschitz stability for the inverse conductivity problem.

Advances in Applied Mathematics, 35(2):207–241.



Calderón, A. P. (1980).

On an inverse boundary value problem.

In *Seminar on Numerical Analysis and Its Applications to Continuum Physics*, pages 65–73, Rio de Janeiro. Brazilian Mathematical Society.

-  Garde, H. and Knudsen, K. (2017).
Distinguishability Revisited: Depth Dependent Bounds on Reconstruction Quality in Electrical Impedance Tomography.
SIAM Journal on Applied Mathematics, 77(2):697–720.
-  Harrach, B. (2023).
The Calderón Problem with Finitely Many Unknowns is Equivalent to Convex Semidefinite Optimization.
SIAM Journal on Mathematical Analysis, pages 5666–5684.
-  Kohn, R. and Vogelius, M. (1984).
Determining conductivity by boundary measurements.
Communications on Pure and Applied Mathematics, 37(3):289–298.

-  Pinilla, S., García, H., Díaz, L., Poveda, J., and Arguello, H. (2018). **Coded aperture design for solving the phase retrieval problem in X-ray crystallography.**
Journal of Computational and Applied Mathematics, 338:111–128.
-  Siltanen, S., Mueller, J., and Isaacson, D. (2000). **An implementation of the reconstruction algorithm of A Nachman for the 2D inverse conductivity problem.**
Inverse Problems, 16(3):681.
-  Sylvester, J. and Uhlmann, G. (1987). **A Global Uniqueness Theorem for an Inverse Boundary Value Problem.**
Annals of Mathematics, 125(1):153–169.

Forward map [Siltanen et al., 2000]

$$\lambda_j(\sigma) = \frac{1 + C_{1,j}}{j\sigma_1(1 - C_{1,j})}$$

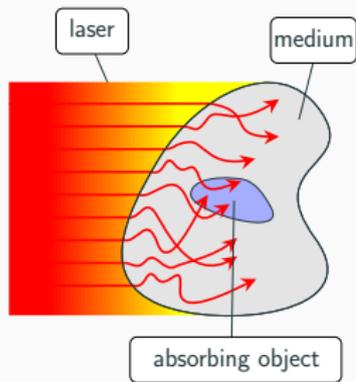
$$C_{n,j} = 0 \quad \text{and} \quad C_{i,j} = \frac{\rho_i r_i^{2j} + C_{i+1,j}}{1 + \rho_i C_{i+1,j} r_i^{-2j}}$$

$$\rho_i = \frac{\sigma_i - \sigma_{i+1}}{\sigma_i + \sigma_{i+1}}$$

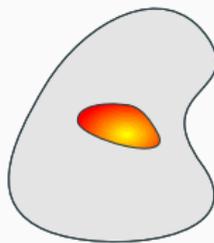
2D case

$$\lambda_j(\sigma) = \frac{1 + \rho_1 r_1^{2j}}{j\sigma_1(1 - \rho_1 r_1^{2j})}$$

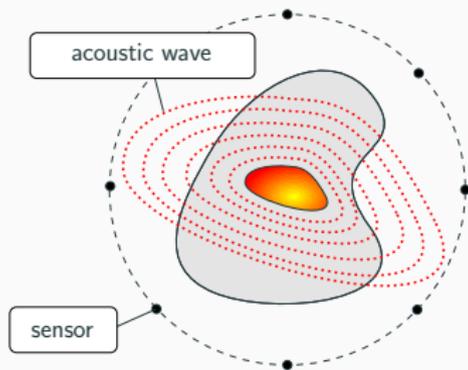
Photo-acoustic tomography



Optical absorption

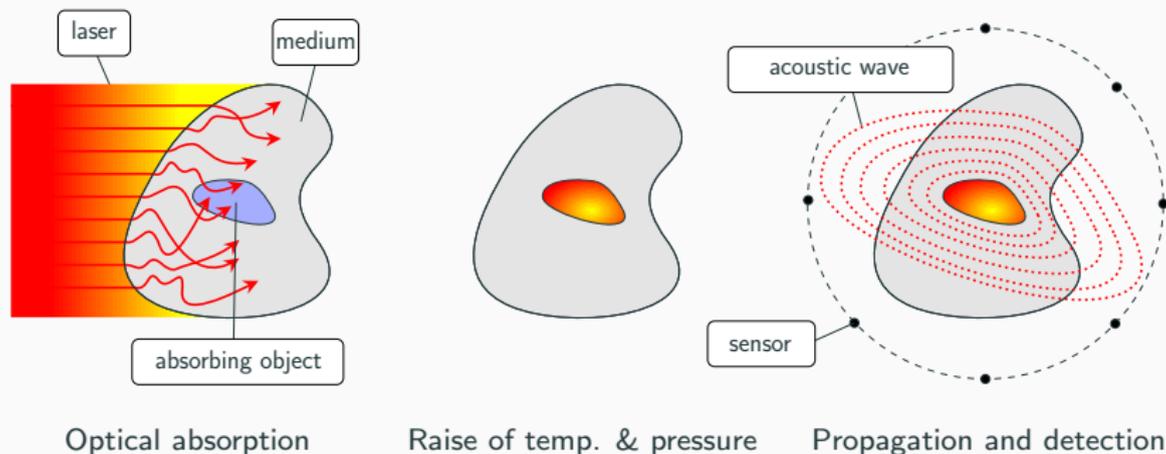


Raise of temp. & pressure



Propagation and detection

Photo-acoustic tomography



Simplified model (2nd step) (see e.g. [Alberti and Capdeboscq, 2018])

- $-\Delta u + qu = 0$ in Ω , $u = g$ on $\partial\Omega$
- Reconstruct q in Ω from u in Ω and q on $\partial\Omega$